Computational and Applied Mathematical Sciences 4 (2): 30-31 2019
ISSN 2222-1328

DOI: 10.5829/idosi.cams.2019.30.31

Identities of Exton-Miller and Paris for $_2F_2$,

[1]J. Yaljá Montiel, [2]J. López-Bonilla and [2]S. Vidal-Beltrán

[1]Centro de Investigación en Computación, Instituto Politécnico Nacional,
Edif. 4, 1er. Piso, Col. Lindavista CP 07738, CDMX, México
[2]ESIME-Zacatenco, Instituto Politécnico Nacional, Edif. 4, 1er. Piso,
Col. Lindavista CP 07738, CDMX, México

Abstract: We show elementary proofs of the identities of Paris and Exton-Miller for the hypergeometric function $_2F_2$.

Key words: Hypergeometric functions, Kummer's identity.

INTRODUCTION

We have the Prudnikov-Bychov-Marichev's formula [1]:

$$ {}_2F_2(a,c+1;\ b,\ c;-x) = {}_1F_1(a,b;-x) - \frac{ax}{bc}\ {}_1F_1(a+1;\ b+1;\ -x), \tag{1}$$

the identity [2]:

$$ {}_1F_1(a-1;b;-x) = {}_1F_1(a;b;-x) + \frac{x}{b}\ {}_1F_1(a;b+1;-x), \tag{2}$$

and the Kummer's transformations [3]:

$$ {}_1F_1(b-a;b;-x) = e^{-x}\ {}_1F_1(a;b;x),\quad {}_1F_1(b-a;b+1;-x) = e^{-x}\ {}_1F_1(a+1;b+1;x), \tag{3}$$

involving the hypergeometric functions $_1F_1$ and $_2F_2$ [4-6].

Now we shall study the following expression:

$$ A \equiv {}_2F_2(b-a-1,\ c+1;\ b,c;-x), \tag{4}$$

in fact:

$$\begin{aligned} A &\overset{(1)}{=} {}_1F_1(b-a-1;\ b;-x) - \frac{(b-a-1)x}{b\ c}\ {}_1F_1(b-a;\ b+1;\ -x), \\ &\overset{(2)}{=} {}_1F_1(b-a;\ b;-x) + \frac{(a-b+c+1)x}{b\ c}\ {}_1F_1(b-a;\ b+1;\ -x), \\ &\overset{(3)}{=} e^{-x}[{}_1F_1(a;\ b;x) + \frac{(a-b+c+1)x}{b\ c}\ {}_1F_1(a+1;\ b+1;\ x)]. \end{aligned} \tag{5}$$

Corresponding Author: J. López-Bonilla, ESIME-Zacatenco, Instituto Politécnico Nacional, Edif. 4, 1er. Piso, Col. Lindavista CP 07738, CDMX, México.

We may consider two interesting cases:

- $c = a - b + 1$. Then from (4) and (5):

$$ {}_2F_2(b-a-1,\ a-b+2;\ b, a-b+1; -x) = e^{-x}\left[{}_1F_1(a;b;x) + \frac{2x}{b}\ {}_1F_1(a+1;\ b+1;\ x) \right], \tag{6} $$

$$ \overset{(1)}{=} e^{-x}\ {}_2F_2\left(a, 1+\frac{a}{2}; b, \frac{a}{2}; x \right), $$

obtained by Exton [7] and Miller [8].

- $c = \dfrac{e(a-b+1)}{a-e}$ Hence from (4) and (5):

$$ {}_2F_2(b-a-1,\ \text{c+1};\ b, c; -x) = e^{-x}\left[{}_1F_1(a;b;x) + \frac{ax}{be}\ {}_1F_1(a+1;\ b+1;\ x) \right], $$

$$ \overset{(1)}{=} e^{-x}\ {}_2F_2\left(a, e+1; b; e; x \right), \tag{7} $$

deduced by Paris [9].

REFERENCES

1. Prudnikov, A.P., Y.A. Bychov and O.I. Marichev, 1990. *Integral and Series, III,* Gordon & Breach, Amsterdam.
2. Slater, L.J., 1966. Generalized hypergeometric functions, Cambridge University Press.
3. Seaborn, J.B., 1991. Hypergeometric functions and their applications, Springer-Verlag, New York.
4. López-Bonilla, J., R. López-Vázquez and S. Vidal-Beltrán, 2018. Identities of Munarini and Simons via Gauss hypergeometric function, European J. Appl. Sci., 10(3): 85-86.
5. López-Bonilla, J., R. López-Vázquez, S. Vidal-Beltrán, 2018. Hypergeometric approach to the Munarini and Ljunggren binomial identities, Comput. Appl. Math. Sci., 3(1): 4-6.
6. Barrera-Figueroa, V., I. Guerrero-Moreno and J. López-Bonilla, 2018. Some applications of hypergeometric functions, Comput. Appl. Math. Sci., 3(2): 23-25.
7. Exton, H., 1997. On the reducibility of the Kampé de Fériet function, J. Comput. Appl. Math., 83: 119-121.
8. Miller, A.R., 2003. On a Kummer-type transformation for the generalized hypergeometric function $_2F_2$, J. Comput. Appl. Math., 157(2): 507-509.
9. Paris, R.B., 2005. A Kummer-type transformation for a $_2F_2$ hypergeometric function, J. Comput. Appl. Math., 173(2): 379-382.