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Chebyshev's Matrices

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Abstract: We exhibit that the Chebyshev's matrices generate an interesting type of associated polynomials which can be generated with the Gauss hypergeometric function and the Bell polynomials.

Key words: Chebyshev polynomials · Gauss hypergeometric function · Characteristic equation

INTRODUCTION

Here we consider the Chebyshev's associated polynomials of degree m in x [1-4]:

$$T_n^m(x) = (-1)^m \binom{2n-m}{m} {}_2F_1\left(-m, 2n-m; n-m+\frac{1}{2}; \frac{1-x}{2}\right), \quad m = 0,1,\ldots,n \tag{1}$$

$$= 2^{m-1}\frac{(n-1)!2(n-m)}{m!(n-m)!}\sum_{k=0}^{m}(-1)^{k-m}\binom{m}{k}\,{}_2F_1(k-m,-1-2m;-2m;1)x^k, \tag{2}$$

defined in terms of the Gauss hypergeometric function ${}_2F_1\,(a, b; c; z)$ [4-6], for example:

$T_3^1 = -5x,\ T_3^2 = 8x^2 - 2,\ T_5^2 = 32x^2 - 4,\ T_5^3 = -56x^3 + 21x,\ T_5^4 = 48x^4 - 36x^2 + 3,$ etc.

verifying the differential equation:

$$(1-x^2)\frac{d^2}{dx^2}T_n^m - (2n-2m+1)x\frac{d}{dx}T_n^m + m(2n-m)T_n^m = 0. \tag{3}$$

The polynomials $T_n^m(x)$ can be interpreted as coefficients in the characteristic equation of the Chebyshev's matrices for a given n [7]:

$$\lambda^n + T_n^1\lambda^{n-1} + T_n^2\lambda^{n-2} + \ldots + T_n^{n-1}\lambda + T_n^n = 0. \tag{4}$$

In Sec. 2 we indicate some properties of the Chebyshev's associated polynomials.

Matrices of Chebyshev: The first-kind Chebyshev polynomials $T_n\,(x)$, $|x| \leq 1$, verify the differential equation [8-18]:

$$(1-x^2)\frac{d^2}{dx^2}T_n - x\frac{d}{dx}T_n + n^2T_n = 0, \quad n = 0,1,2,\ldots \tag{5}$$

which is equivalent to the following expression in terms of the Gauss hypergeometric function [4-6]:

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$$T_n(x) = {}_2F_1\left(-n,\ n; \frac{1}{2}; \frac{1-x}{2}\right). \tag{6}$$

The equation (5) is obtained from (3) for the case $m = n$, then (1) and (6) imply the following connection for the first-kind; $T_n(x) = (-1)^n\, T_n^n(x)$; besides, it is easy to show that the associated polynomials (1) can generate the other types of Chebyshev polynomials [15, 18]:

$$U_n(x) = \frac{2(-1)^n}{n+2} T_{n+1}^n(x),\quad V_n(x) = \frac{(-1)^n}{n+1} T_{2n+1}^{2n}\left(\sqrt{\frac{1-x}{2}}\right),\quad W_n(x) = \frac{1}{n+1} T_{2n+1}^{2n}\left(\sqrt{\frac{1+x}{2}}\right); \tag{7}$$

for example, (3) with $n = m + 1$ implies the differential equation satisfied by the Chebyshev polynomials of the second kind:

$$(1-x^2)\frac{d^2}{dx^2}U_m - 3x\frac{d}{dx}U_m + m(m+2)U_m = 0,\quad m = 0,1,2,... \tag{8}$$

We have the Chebyshev tridiagonal matrices [7]:

$$\tilde{T}_1 = (x),\quad \tilde{T}_2 = \begin{pmatrix} x & 1 \\ 1 & 2x \end{pmatrix},\quad \tilde{T}_3 = \begin{pmatrix} x & 1 & 0 \\ 1 & 2x & 1 \\ 0 & 1 & 2x \end{pmatrix},\quad \tilde{T}_4 = \begin{pmatrix} x & 1 & 0 & 0 \\ 1 & 2x & 1 & 0 \\ 0 & 1 & 2x & 1 \\ 0 & 0 & 1 & 2x \end{pmatrix},... \tag{9}$$

whose determinant gives the corresponding Chebyshev polynomial of the first kind:

$$T_n(x) = \det \tilde{T}_n \therefore T_1 = x,\quad T_2 = 2x^2 - 1,\quad T_3 = 4x^3 - 3x,\quad T_4 = 8x^4 - 8x^2 + 1,... \tag{10}$$

and the characteristic equations of (9) have the structure (4), hence the Chebyshev's associated polynomials can be written in terms of the Bell polynomials [19-27]:

$$T_n^m = \frac{1}{m!} Y_m(-0!s_1, -1!s_2, -2!s_3, 3!s_4, ..., -(m-2)!s_{m-1}, -(m-1)!s_m), \tag{11}$$

such that:

$$s_r = trace\left[\tilde{T}_n\right]^r,\quad r = 1,...,m, \tag{12}$$

that is, the Bell polynomials and the traces of the powers of Chebyshev's matrices (9) allow construct the associated polynomials (1).

REFERENCES

1. Guerrero-Moreno, I., J. López-Bonilla and L. Rosales, 2008. On the associated polynomials of Chebyshev, Bol. Soc. Cub. Mat. Comp., 6(2): 93-97.
2. López-Bonilla, J., M. Turgut and A. Zaldívar-Sandoval, 2011. Associated polynomials of Chebyshev, Dhaka Univ. J. Sci., 59(1): 153-154.
3. Barrera-Figueroa, V., R. Cruz-Santiago and J. López-Bonilla, 2018. Chebyshev's associated polynomials, European J. Appl. Sci., 10(3): 87-88.
4. López-Bonilla, J., R. López-Vázquez and J.C. Prajapati, 2014. Some applications of ${}_2F_1$ (a, b; c; z), Int. J. Mathematical Engineering and Science, 3(1): 41-46.
5. Seaborn, J.B., 1991. Hypergeometric functions and their applications, Springer-Verlag, New York.
6. Forrey, R.C., 1997. Computing the hypergeometric function, J. Comput. Phys., 137: 79-100.
7. Álvarez-Ballesteros, S., A. Bucur and J. López-Bonilla, 2007. On the characteristic equation of Chebyshev matrices, General Mathematics, 15(4): 17-23.

8. Chebyshev, P.L., 1854. Théorie des mécanismes connus sous le nom de parallélogrammes, Mém. Acad. Sci. Pétersb., 7: 539-568.
9. Lanczos, C., 1938. Trigonometric interpolation of empirical and analytical functions, J. Math. Phys. 17: 123-199.
10. Lanczos, C., 1952. Chebyshev polynomials in the solution of large-scale linear systems, Toronto Symposium on Computing Techniques, pp: 124-133.
11. Lanczos, C., 1952. Tables of Chebyshev polynomials, Nat. Bur. Std. Appl. Math. Series No. 9:
12. Lanczos, C., 1973. Legendre versus Chebyshev polynomials, in 'Topics in numerical analysis' (Proc. Roy. Irish Acad. Conf. on Numerical Analysis, Aug. 14-18, 1972), Academic Press, London, pp: 191-201.
13. Rivlin, T.J., 1974. The Chebyshev polynomials, Wiley-Interscience, New York.
14. Lanczos, C., 1988. Applied analysis, Dover, New York.
15. Mason, J.C. and D. C. Handscomb, 2002. Chebyshev polynomials, Chapman & Hall / CRC, London.
16. Piessens, R., 2010. Applications of Chebyshev polynomials: from theoretical kinematics to practical computations, in "The birth of numerical analysis", Eds. A. Bultheel, R. Cools, World Scientific, Singapore, pp: 193-205.
17. Cruz-Santiago, R., C. Hernández-Aguilar and J. López-Bonilla, 2018. An application of the tau method, Comput. & Appl. Math. Sci., 3(2): 15-17.
18. Lam-Estrada, P., J. López-Bonilla and R. López-Vázquez, 2015. Baldoni et al method for homogeneous linear recurrence relations, Proc. Int. Conf. on Special Functions & Applications, Amity University, Noida, Uttar Pradesh, India, Sept. 10-12, 2015.
19. Riordan, J., 1968. Combinatorial identities, John Wiley and Sons, New York.
20. Comtet, L, 1974. Advanced combinatorics, D. Reidel Pub., Dordrecht, Holland.
21. Zave, D.A., 1976. A series expansion involving the harmonic numbers, Inform. Process. Lett., 5(3): 75-77.
22. Chen, X. and W. Chu, 2009. The Gauss -summation theorem and harmonic number identities, Integral Transforms and Special Functions, 20(12): 925-935.
23. Quaintance, J. and H.W. Gould, 2016. Combinatorial identities for Stirling numbers, World Scientific, Singapore.
24. Johnson, W.P., 2002. The curious history of Faà di Bruno's formula, The Math. Assoc. of America, 109: 217-234.
25. López-Bonilla, J., R. López-Vázquez and S. Vidal-Beltrán, 2018. Bell polynomials, Prespacetime, 9(5): 451-453.
26. López-Bonilla, J., S. Vidal-Beltrán and A. Zúñiga-Segundo, 2018. Characteristic equation of a matrix via Bell polynomials, Asia Mathematika, 2(2): 49-51.
27. López-Bonilla, J., R. López-Vázquez and S. Vidal-Beltrán, 2019. Applications of Bell polynomials, World Eng. & Appl. Sci. J., 10(1): 4-7.