Computational and Applied Mathematical Sciences 4 (2): 49-50 2019
ISSN 2222-1328

DOI: 10.5829/idosi.cams.2019.49.50

An Elementary Proof of the Padoa's Inequality

[1]J. Yaljá Montiel-Pérez, [2]R. Cruz-Santiago and [2]J. López-Bonilla

[1]Centro de Investigación en Computación, Instituto Politécnico Nacional
[2]ESIME-Zacatenco, Instituto Politécnico Nacional, Edif. 5, 1er. Piso,
Col. Lindavista CP 07738, CDMX, México

Abstract: We give a simple proof of the inequality of Padoa involving the sides of an arbitrary triangle.

Key words: Euler's relation · Geometry of triangles · Heron's formula

INTRODUCTION

Nahin [1] proposes to give a proof of the Padoa's inequality:

$$(a + b - c)(b + c - a)(a + c - b) \leq abc, \tag{1}$$

involving the sides of an arbitrary triangle.

In [2, 3] we find the following upper bound on the area $\tilde{A}$ of a triangle:

$$4\sqrt{3}(a+b+c)\,\tilde{A} \leq 9\; a\; b\; c, \tag{2}$$

where we can employ the Euler's formula:

$$\tilde{A} = \frac{a\; b\; c}{4R}, \tag{3}$$

in terms of the circumradius, to obtain:

$$2\sqrt{3}\; s \leq 9\; R, \tag{4}$$

for the semiperimeter $s = \frac{1}{2}(a+b+c);$ but we know the relations:

$$s = a\frac{Cos\left(\frac{B}{2}\right)Cos\left(\frac{C}{2}\right)}{Sin\left(\frac{A}{2}\right)}, \qquad R = \frac{a}{2\; Sin\; A}, \tag{5}$$

then (4) implies the property:

$$Cos\left(\frac{A}{2}\right)Cos\left(\frac{B}{2}\right)Cos\left(\frac{C}{2}\right) \leq \frac{3}{8}\sqrt{3}, \tag{6}$$

Corresponding Author: J. López-Bonilla, ESIME-Zacatenco, Instituto Politécnico Nacional, Edif. 5, 1er. Piso, Col. Lindavista CP 07738, CDMX, México.

On the other hand, in [3-5] is the inequality:

$$3\sqrt{3}\ \tilde{A} \leq s^2, \tag{7}$$

where we can apply the expressions involving the inradius:

$$\tilde{A} = r\ s, \qquad r = s\ .\ tan\left(\frac{A}{2}\right)tan\left(\frac{B}{2}\right)tan\left(\frac{C}{2}\right) = a\frac{sin\left(\frac{B}{2}\right)sin\left(\frac{C}{2}\right)}{Cos\left(\frac{A}{2}\right)}, \tag{8}$$

to deduce that:

$$tan\left(\frac{A}{2}\right)tan\left(\frac{B}{2}\right)tan\left(\frac{C}{2}\right) \leq \frac{1}{3\sqrt{3}}, \tag{9}$$

therefore, (6) and (9) give the relations:

$$Sin\left(\frac{A}{2}\right)Sin\left(\frac{B}{2}\right)Sin\left(\frac{C}{2}\right) \leq \frac{1}{8}, \qquad Cos\ A + Cos\ B + Cos\ C \leq \frac{3}{2}; \tag{10}$$

with (5), (8) and (10) we obtain the inequality:

$$R \geq 2\ r, \tag{11}$$

Now we employ the Heron's formula:

$$(s-a)(s-b)(s-c) = \frac{A^2}{s} \underset{=}{(8)}\ r\ s\ .\ r \underset{=}{(3)} \frac{a\ b\ c}{4} * \frac{r}{R} \leq \frac{a\ b\ c}{8}, \tag{12}$$

which implies the Padoa's inequality (1) because:

$$s-a = \frac{1}{2}(b+c-a), \qquad s-b = \frac{1}{2}(a+c-b), \qquad s-c = \frac{1}{2}(a+b-c). \tag{13}$$

REFERENCIAS

1. Nahin, P.J., 2004. When Least is Best, Princeton University Press (2004).
2. Posamentier, A.S. and I. Lehmann, 2012. The Secrets of Triangles: A Mathematical Journey, Prometheus Books, Amherst, New York.
3. https://en.wikipedia.org/wiki/Triangle
4. Chakerian, G.D., 1979.. A distorted view of geometry, in 'Mathematical plums', Ed. R. Honsberger, Math. Assoc. Am., Washington, D.C. (1979) Chap. 7.
5. Rosenberg, S., M. Spillane and D.B. Wulf, 2008. Heron triangles and moduli spaces, Maths. Teacher, 101: 656-663.