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Darboux transformation

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Abstract: We show elementary approaches to Darboux mapping applied to Schrödinger equation.

Key words: Isospectral potentials · Darboux transform

INTRODUCTION

In the one-dimensional stationary case the Schrödinger equation is given by [1]:

$$-\frac{d^2}{dx^2}\psi + u(x)\psi = \lambda\psi, \tag{1}$$

in natural units such that $\frac{\hbar^2}{2m} = 1$. The values of λ represent the energy spectrum allowed for certain boundary conditions and corresponding to the standard potential $u(x)$

The Darboux transform (DT) [2-10] allows construct a new Schrödinger equation:

$$-\frac{d^2}{dx^2}\phi + U(x)\phi = \lambda\phi, \tag{2}$$

for another potential $U(x)$ with the same energy levels, hence u and U are isospectral potentials. In the next Sec. 2 we deduce, in natural manner, the principal expressions of the DT associated to the particular solution ψ_1 with eigenvalue λ_1

$$-\psi_1'' + u\psi_1 = \lambda_1\psi_1, \tag{3}$$

thus we say that the 'seed function' ψ_1 generates a DT. Selecting diverse seed functions we can obtain many DTs and therefore a great variety of generalized potentials with the same energy spectrum; that is, the Schrödinger equation is covariant with respect to Darboux transform.

In Sec. 3 we deduce again the DT via an adequate factorization of the Hamiltonian involved in Schrödinger equation.

Darboux Mapping: We use ψ_1 to introduce into (1) the factorization:

$$\psi = \psi_1\,\beta, \tag{4}$$

then $\beta(x)$ satisfies the relation:

$$\beta'' + 2\sigma_1\beta' + (\lambda - \lambda_1)\beta = 0, \qquad \sigma_1 \equiv \frac{\psi_1'}{\psi_1}, \tag{5}$$

and now to (5) we apply $\frac{d}{dx}$ to obtain a differential equation of second order for β:

$$\frac{d^2}{dx^2}\beta' + 2\sigma_1\frac{d}{dx}\beta' + (\lambda - \lambda_1 + 2\sigma_1')\beta' = 0, \tag{6}$$

which can be transformed to the Schrödinger equation:

$$-\phi'' + (\lambda_1 - \sigma_1' + \sigma_1^2)\phi = \lambda\phi, \tag{7}$$

via the mapping [11]:

$$\phi = \psi_1\beta' \overset{(4)}{=} \psi_1\frac{d}{dx}\left(\frac{\psi}{\psi_1}\right) \overset{(5)}{=} \psi' - \sigma_1\psi, \tag{8}$$

and (7) has the structure (2) for the new isospectral potential:

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$$U(x) = \lambda_1 - \sigma_1^{'} + \sigma_1^2 = u - 2\sigma_1^{'}, \tag{9}$$

because (3) implies the connection $u = \lambda_1 + \sigma_1^{'} + \sigma_1^2.$

Factorization of the Hamiltonian: The equation (1) means $H\psi = \lambda\psi$ such that:

$$H \equiv -\frac{d^2}{dx^2} + u(x), \tag{10}$$

and we search operators of first order [1, 12, 13]:

$$A = \frac{d}{dx} + W(x), \qquad A^{\dagger} = \frac{d}{dx} + W(x), \tag{11}$$

to factorize (10) in the form:

$$H \equiv A^{\dagger}A + \lambda_1 = -\frac{d^2}{dx^2} + W^2 - W^{'} + \lambda_1, \tag{12}$$

and its comparison with (10) implies the relationship:

$$u = W^2 - W^{'} + \lambda_1, \tag{13}$$

From (3) we have that $u = \lambda_1 \frac{\psi_1^{"}}{\psi_1},$ hence (13) gives the following Riccati equation [9, 11, 14-16] for the superpotential $W(x)$

$$W^{'} - W^2 + \frac{\psi_1^{"}}{\psi_1} = 0, \tag{14}$$

and under the change of variable [11] $W = -\frac{\eta^{'}}{\eta}$ it is reduced to $\frac{\eta^{"}}{\eta} = \frac{\psi_1^{"}}{\psi_1},$, therefore $\eta = \psi_1$, then:

$$W = -\frac{\psi_1^{'}}{\psi_1} = -\sigma_1, \tag{15}$$

in according with (5), and (13) gives the known relation $u = +\lambda_1 - \sigma_1^{'} + \sigma_1^2.$ From (11) and (15):

$$A = \frac{d}{dx} - \sigma_1, \qquad A^{\dagger} = -\frac{d}{dx} - \sigma_1 \qquad \therefore \quad \mathrm{A}\psi_1 = 0. \tag{16}$$

Now we construct the Hamiltonian:

$$\tilde{H} \equiv AA^{\dagger} + \lambda_1 = -\frac{d^2}{dx^2} + \sigma_1^2 - \sigma_1^{'} + \lambda_1 = -\frac{d^2}{dx^2} + U(x), \tag{17}$$

in harmony with (9) and we ask that H and $\tilde{H}$ have the same energy levels, then if the eigenvalue expression $H\psi = A^{\dagger}A\psi + \lambda_1\psi = \lambda\psi$ is multiplied by A results:

$$(AA^{\dagger} + \lambda_1)A\psi \equiv \tilde{H}A\psi = \lambda A\psi,$$

hence the corresponding wave functions of $\tilde{H}$ are given by $\phi = A\psi = \left(\frac{d}{dx} - \sigma_1\right)\psi = \psi^{'} - \sigma_1\psi,$ which coincides with (8).

The expressions (8) and (9) represent the Darboux transformation based in the seed function ψ_1 Our processes are explicit motivations for this important mapping of the mathematical physics.

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