

Lanczos Approach to Noether Theorem and Genuine Constraints

¹J. Yaljá Montiel-Pérez, ²J. López-Bonilla and ²R. López-Vázquez

¹Centro de Investigación en Computación, Instituto Politécnico Nacional, CDMX, México

²ESIME-Zacatenco, Instituto Politécnico Nacional, Edif. 4,
 1er. Piso, Col. Lindavista CP 07738, CDMX, México

Abstract: For several singular Lagrangians we introduce one arbitrary function by each effective gauge parameter to employ the Lanczos approach of the Noether theorem and thus to exhibit the corresponding genuine constraints and their conservation by the time evolution.

Key words: Genuine constraints • Singular Lagrangians • Constrained Hamiltonian systems

INTRODUCTION

We apply the matrix technique to several Lagrangians studied in [1-3] and we employ the Lanczos approach [4-6] to Noether's theorem for each effective gauge parameter and we observe that the corresponding Euler-Lagrange equations are in terms of the genuine constraints and their time evolution. To save comments and notations it will be evident when certain quantities are satisfied on shell. Here we consider the following four Lagrangians whose matrix analysis allows show the application of the mentioned Lanczos procedure:

$$\text{a). Ref. [7]: } L = \frac{1}{2} \dot{q}_1^2 + \dot{q}_1 q_2 + \frac{1}{2} (q_1 - q_2)^2, \quad N = 2. \quad (1)$$

The Lagrangian method [1-3, 7] gives the Hessian matrix $w^{(0)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ whose rank is 1, with only one genuine constraint and one gauge identity:

$$\varphi^{(0,1)} = -q_2 - \dot{q}_1 + q_1, \quad E_1^{(0)} + E_2^{(0)} + \frac{d}{dt} E_2^{(0)} = 0, \quad (2)$$

and the corresponding local gauge symmetries have the following structure:

$$\tilde{q}_1 = q_1 + \varepsilon a, \quad \tilde{q}_2 = q_2 + \varepsilon (a - \dot{a}), \quad \varepsilon \ll 1, \quad (3)$$

where a is an arbitrary function, then we have to a and \dot{a} as effective gauge parameters.

The essence of the Lanczos technique [4-6], to obtain information from the invariance of L under an infinitesimal transformation, consists in to consider the effective gauge parameters as new degrees of freedom for the variational problem under study. Then we can apply this idea of Lanczos to the local gauge transformation (3), in fact, we introduce the functions $\beta_1 = a$ and $\beta_2 = \dot{a}$ as new degrees of freedom, thus (3) takes the form $\tilde{q}_1 = q_1 + \varepsilon \beta_1$ & $\tilde{q}_2 = q_2 + \varepsilon (\beta_1 - \beta_2)$ producing the following change in the Lagrangian for $\varepsilon \ll 1$.

$$\tilde{L} = L + \varepsilon [\beta_1 \dot{q}_1 + \dot{\beta}_1 (q_2 + \dot{q}_1) + \beta_2 (-\dot{q}_1 + q_1 - q_2)], \quad (4)$$

and the corresponding Euler-Lagrange equations $\frac{d}{dt}\left(\frac{\partial L}{\partial \beta_r}\right) - \frac{\partial L}{\partial \beta_r} = 0$, $r = 1, 2$ imply the relations:

$$\varphi^{(0,1)} = 0, \quad \frac{d}{dt}\varphi^{(0,1)} = 0, \quad (5)$$

in terms of the genuine constraint and its conservation by the time evolution. In other words, the Lanczos approach shows that the effective gauge parameters store information about the genuine constraints and their time derivative. Besides, let's remember that in [2] was exhibited the connection between the genuine constraints and the non-primary constraints (that is, secondary, tertiary,...) for the Lagrangians considered in the present work.

b). Ref. [8]:
$$L = \frac{1}{2}[(\dot{q}_2 - e^{q_1})^2 + (\dot{q}_3 - q_2)^2], \quad N = 3. \quad (6)$$

The matrix procedure and (6) lead to $W^{(0)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ such that $rank W^{(0)} = 2$, with one gauge identity and two genuine constraints:

$$\begin{aligned} \dot{q}_1^2 E_1^{(0)} + \dot{q}_1 e^{q_1} E_2^{(0)} + e^{q_1} E_3^{(0)} - \dot{q}_1 \frac{d}{dt} E_1^{(0)} + \frac{d}{dt} \left(-\dot{q}_1 E_1^{(0)} - e^{q_1} E_2^{(0)} + \frac{d}{dt} E_1^{(0)} \right) &= 0, \\ \varphi^{(0,1)} &= e^{q_1} (\dot{q}_2 - e^{q_1}), \quad \varphi^{(1,1)} = e^{q_1} (q_2 - \dot{q}_3), \end{aligned} \quad (7)$$

and the gauge transformations take the form:

$$\tilde{q}_1 = q_1 + \varepsilon \dot{\alpha} e^{-q_1}, \quad \tilde{q}_2 = q_2 + \varepsilon \dot{\alpha}, \quad \tilde{q}_3 = q_3 + \varepsilon \alpha, \quad (8)$$

with α , $\dot{\alpha}$ and $\ddot{\alpha}$ as effective gauge parameters.

Now we employ the functions $\beta_1 = \alpha$, $\beta_2 = \dot{\alpha}$ and $\beta_3 = \ddot{\alpha}$ to implement the Lanczos process via (8) with the structure $\tilde{q}_1 = q_1 + \varepsilon e^{-q_1} \beta_3$, $\tilde{q}_2 = q_2 + \varepsilon \beta_2$ & $\tilde{q}_3 = q_3 + \varepsilon \beta_1$, generating thus a functional change in L :

$$\tilde{L} = L + \varepsilon e^{-q_1} [\varphi^{(0,1)} (\dot{\beta}_2 - \beta_3) - \varphi^{(1,1)} (\dot{\beta}_1 - \beta_2)], \quad (9)$$

and the Euler-Lagrange equations for β_r , $r = 1, 2, 3$ give the conditions:

$$\varphi^{(0,1)} = 0, \quad \frac{d}{dt}\varphi^{(0,1)} - \dot{q}_1 \varphi^{(0,1)} + \varphi^{(1,1)} = 0, \quad \frac{d}{dt}\varphi^{(0,1)} - \dot{q}_1 \varphi^{(1,1)} = 0, \quad (10)$$

which is information about the constraints and their time evolution.

c). Refs. [9, 10]:
$$L = (\dot{q}_1 + q_2)\dot{q}_3 + q_1 q_3, \quad N = 3 \quad (11)$$

Now $rank W^{(0)} = 2$, with two genuine constraints and one gauge identity:

$$\varphi^{(0,1)} = \dot{q}_3, \quad \varphi^{(1,1)} = -q_3, \quad E_2^{(0)} + \frac{d}{dt} \left(E_1^{(0)} - \frac{d}{dt} E_2^{(0)} \right) = 0, \quad (12)$$

and the local transformations are given by:

$$\tilde{q}_1 = q_1 + \varepsilon \dot{\alpha}, \quad \tilde{q}_2 = q_2 + \varepsilon(\ddot{\alpha} - \alpha), \quad \tilde{q}_3 = q_3, \quad (13)$$

with the presence of α and its two derivatives, hence we use the functions $\beta_1 = \alpha$, $\beta_2 = \dot{\alpha}$ and $\beta_3 = \ddot{\alpha}$ to implement the Lanczos method via (13) with the structure $\tilde{q}_1 = q_1 + \varepsilon e^{-q_1} \beta_3$, $\tilde{q}_2 = q_2 + \varepsilon \beta_2$ & $\tilde{q}_3 = q_3 + \varepsilon \beta_1$, generating thus a functional change in L :

$$\tilde{L} = L + \varepsilon e^{-q_1} [\varphi^{(0,1)}(\dot{\beta}_2 - \beta_3) - \varphi^{(1,1)}(\dot{\beta}_1 - \beta_2)], \quad (14)$$

and the Euler-Lagrange equations for β_r , $r = 1, 2, 3$ give the conditions:

$$\varphi^{(0,1)} = 0, \quad \frac{d}{dt} \varphi^{(0,1)} - \dot{q}_1 \varphi^{(0,1)} + \varphi^{(1,1)} = 0, \quad \frac{d}{dt} \varphi^{(1,1)} - \dot{q}_1 \varphi^{(1,1)} = 0, \quad (15)$$

in terms of the corresponding genuine constraints and their time evolution.

$$\text{d). Ref. [7]:} \quad L = \frac{1}{2} \dot{q}_1^2 + (q_2 - q_3) \dot{q}_1 + \frac{1}{2} (q_1 - q_2 + q_3)^2, \quad N = 3 \quad (16)$$

Here $\text{rank } W^{(0)} = 1$, with two gauge identities and one genuine constraint:

$$\varphi^{(0,1)} = -\dot{q}_1 + q_1 - q_2 + q_3, \quad E_2^{(0)} + E_3^{(0)} = 0, \quad E_1^{(0)} + E_2^{(0)} + \frac{d}{dt} E_2^{(0)} = 0, \quad (17)$$

participating the gauge transformations:

$$\tilde{q}_1 = q_1 + \varepsilon \alpha, \quad \tilde{q}_2 = q_2 + \varepsilon(\alpha - \dot{\alpha} + \gamma), \quad \tilde{q}_3 = q_3 + \varepsilon \gamma, \quad (18)$$

where $\dot{\alpha}$, α and γ are the effective gauge parameters. Here we employ (18) with the functions $\beta_1 = \alpha$, $\beta_2 = \dot{\alpha}$ and $\beta_3 = \gamma$, that is, $\tilde{q}_1 = q_1 + \varepsilon \beta_1$, $\tilde{q}_2 = q_2 + \varepsilon(\beta_1 - \dot{\beta}_1 + \beta_3)$ & $\tilde{q}_3 = q_3 + \varepsilon \beta_3$, therefore:

$$\tilde{L} = L + \varepsilon [(q_1 - \varphi^{(0,1)}) \dot{\beta}_1 - \dot{q}_1 \beta_1 + \varphi^{(0,1)} \beta_2], \quad (19)$$

and the Euler-Lagrange equations for the new degrees of freedom β , imply (5) with $\varphi^{(0,1)}$ given in (17).

The invariance of L under local gauge transformations leads to constraints, then it is natural that the effective gauge parameters store information about the genuine constraints if they are considered as additional degrees of freedom for the variational problem under analysis; the Lanczos approach allows extract such information.

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