

An Identity Involving Stirling Numbers

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Abstract: We employ the results of Xu-Cen and Gould to deduce an identity for Stirling numbers.

Key words: Stirling numbers • Gould and Xu-Cen identities

INTRODUCTION

Our study is related with the expression:

$$\sum_{k=j}^n (-1)^{n-k} \binom{k+\lambda}{j} k! S_n^{[k]} \quad (1)$$

where the case $\lambda = -1$ was considered by Gould [1, 2]:

$$\sum_{k=j}^n (-1)^{n-k} \binom{k-1}{j} k! S_n^{[k]} = (j+1)! S_n^{[j+1]}, \quad j = 0, 1, \dots, n; \quad n \geq 1, \quad (2)$$

and Xu-Cen [3] worked the case $\lambda = 0$:

$$\sum_{k=j}^n (-1)^{n-k} \binom{k}{j} k! S_n^{[k]} = j! S_{n+1}^{[j+1]}, \quad 0 \leq j \leq n, \quad n \geq 0, \quad (3)$$

In Sec. 2, for $\lambda = 1$ we obtain the relation:

$$\sum_{k=j}^n (-1)^{n-k} \binom{k+1}{j} k! S_n^{[k]} = (j-1)! \left[S_{n+2}^{[j+1]} - S_{n+1}^{[j+1]} + (-1)^{n-j} S_n^{[j-1]} \right], \quad j = 1, \dots, n. \quad (4)$$

An Identity for Stirling Numbers:

In fact, from (2):

$$\begin{aligned} (j+1)! S_{n+1}^{[j+1]} &= \sum_{k=j+1}^{n+1} (-1)^{n+1-k} \binom{k-1}{j} k! S_{n+1}^{[k]} = \sum_{k=j}^n (-1)^{n+1-k} \binom{k-1}{j} k! S_{n+1}^{[k]} + \binom{n}{j} (n+1)! \\ &= \binom{n}{j} (n+1)! - \sum_{k=j}^n (-1)^{n-k} \binom{k-1}{j} k! \left[S_n^{[k-1]} + k S_n^{[k]} \right], \end{aligned}$$

that is:

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$$j! S_{n+1}^{[j+1]} = n! \binom{n+1}{j+1} - \sum_{k=j}^n (-1)^{n-k} \binom{k}{j+1} k! S_n^{[k]} - \sum_{k=j+1}^n (-1)^{n-k} \binom{k}{j+1} (k-1)! S_n^{[k-1]},$$

where we can use (3) and the recurrence relation for Stirling numbers [2]:

$$S_{r+1}^{[m]} = S_r^{[m-1]} + m S_r^{[m]}, \quad (5)$$

to deduce the expression:

$$\sum_{k=j+1}^n (-1)^{n-k} \binom{k}{j+1} (k-1)! S_n^{[k-1]} = n! \binom{n+1}{j+1} - j! [S_{n+2}^{[j+2]} - S_{n+1}^{[j+2]}], \quad (6)$$

which is equivalent to (4), q.e.d.

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