

On a Lagrangian for the Linear Differential Equation of 2nd Order

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Abstract: The differential equation $py'' + qy' + ry = \phi$ is the Euler-Lagrange equation of the variational principle $\delta \int_{x_1}^{x_2} L dx = 0$ for certain Lagrangian L . If $y_1(x)$ is a solution of the corresponding homogeneous differential equation, then the invariance of L under the transformation $y(x) \rightarrow y(x) + \varepsilon y_1(x)$, $\varepsilon \ll 1$, gives the complete solution of the original non-homogeneous equation, in harmony with the method of variation of parameters.

Key words: 2th order linear differential equation • Variation of parameters • Noether's theorem

INTRODUCTION

For the differential equation:

$$p y'' + qy' + ry = \phi, \quad (1)$$

We have the solution $y_1(x)$ of the corresponding homogeneous equation:

$$py_1'' + qy_1' + ry_1 = 0; \quad (2)$$

then it is known that the complete integration of (1) is given by [1-4]:

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + y_p(x), \quad (3)$$

where y_2 is other independent solution of (2):

$$y_2(x) = y_1(x) \int^x \frac{W}{y_1^2} d\xi, \quad W(x) = \exp\left(-\int^x \frac{q}{p} d\eta\right), \quad (4)$$

and y_p is the particular solution of (1):

$$y_p(x) = y_2(x) \int^x \frac{y_1 \phi}{pw} d\xi - y_1(x) \int^x \frac{y_2 \phi}{p w} d\xi. \quad (5)$$

The variational principle $\delta \int_{x_1}^{x_2} L dx = 0$ with the Lagrangian [5-7]:

$$L = \frac{1}{2} \left(y'^2 - \frac{r}{p} y^2 + 2 \frac{\phi}{p} y \right) \exp\left(\int^x \frac{q}{p} d\xi\right), \quad (6)$$

gives (1). In Sec. 2 we employ the Noether theorem [8-13] to show that the invariance of (6) under the transformation $y(x) \rightarrow y(x) + \varepsilon y_1(x)$ implies (3, 4, 5).

Noether's Theorem: We accept that L is invariant under the transformation:

$$\tilde{x} = x, \quad \tilde{y}(x) = y(x) + \varepsilon y_1(x), \quad \varepsilon = \text{constant} \ll 1, \quad (7)$$

then the Noether theorem, in the approach of Lanczos [8, 9, 12, 13], indicates apply (7) to (6) with $\varepsilon(x)$ and after to write the Euler-Lagrange equation associated to ε , in fact:

$$\tilde{L} = \frac{1}{2w} \left(\tilde{y}'^2 - \frac{r}{p} \tilde{y}^2 + 2 \frac{\phi}{p} \tilde{y} \right) = L + \frac{2}{w} \left[\varepsilon' y_1 y' + \varepsilon \left(y_1' y' - \frac{r}{p} y_1 y + \frac{\phi}{p} y_1 \right) \right],$$

then the condition $\frac{d}{dx} \left(\frac{\partial L}{\partial \varepsilon'} \right) - \frac{\partial L}{\partial \varepsilon} = 0$ leads to:

$$\begin{aligned} \frac{y_1 \phi}{pw} &= \frac{d}{dx} \left(\frac{y_1 y'}{w} \right) - \frac{1}{w} \left(y_1' y' - \frac{r}{p} y_1 y \right)^{(2)} = \frac{d}{dx} \left(\frac{y_1 y'}{w} \right) - \frac{1}{w} \left[y_1' y' + \frac{y}{p} (p y_1'' + q y_1') \right], \\ &= \frac{d}{dx} \left[y_1 y' \exp \left(\int^x \frac{q}{p} d\xi \right) \right] + \left[\frac{d}{dx} (-y_1' y) \right] \exp \left(\int^x \frac{q}{p} d\xi \right) + (-y_1' y) \frac{d}{dx} \exp \left(\int^x \frac{q}{p} d\xi \right), \\ &= \frac{d}{dx} \left[(y_1 y' - y_1' y) \exp \left(\int^x \frac{q}{p} d\xi \right) \right], \end{aligned} \quad (8)$$

then the integration of (8) allows to deduce (3, 4, 5) q.e.d.

Hence the known expressions for $y_2(x)$ and $y_p(x)$ are consequences of the symmetry of (6) under the transformation (7) generated by $y_1(x)$

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