

Some Propagators in Quantum Mechanics

¹*A. Hernández-Galeana, ¹J. Rivera-Rebolledo, ²J. López-Bonilla and ²R. López-Vázquez*

¹ESFM, Instituto Politécnico Nacional, Depto. Física, Edif. 9,
 Col. Lindavista CP 07738, CDMX

²ESIME-Zacatenco, Instituto Politécnico Nacional, Edif. 4, 1er. Piso,
 Col. Lindavista CP 07738, CDMX, México

Abstract: We consider one-dimensional non-relativistic quantum mechanics to exhibit that the propagators for free particle, linear potential and harmonic oscillator, are obtainable from purely classical means.

Key words: Green functions • Propagators in quantum mechanics

INTRODUCTION

The wave function $\psi(x, t)$ satisfies the Schrödinger equation [1]:

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2 \partial^2 \psi}{2m \partial x^2} + V(x)\psi, \quad (1)$$

which accepts solution via Green's technique [2]:

$$\psi(x, t) = \int_{-\infty}^{\infty} K(x, x', t)\psi(x', 0)dx', \quad (2)$$

where $K(x, x', t)$ is the corresponding propagator for the potential $V(x)$ with the properties:

$$\left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x'^2} + V(x) - i\hbar \frac{\partial}{\partial t} \right) K(x, x', t) = 0, \\ \lim_{t \rightarrow 0} K(x, x', t) = \delta(x - x'), \quad (3)$$

In according with Dirac [3, 4] and Feynman [5-7]:

$$K(x, x', t) \propto \exp\left(\frac{i}{\hbar} \int_0^t L dt\right), \quad (4)$$

being L the Lagrangian of the system under study. In (4) are important all paths with initial and final points $(x', 0)$ and (x, t) respectively; in this connection we remember the comment from Sommerfeld [8]: ‘The causality of the 20th century must not limit itself to the initial state, but must take end-state into consideration as an equally decisive moment’.

Here we determine the classical action integral $\int_0^t L dt$ associated with the path actually traversed between the points $(x', 0)$ and (x, t) and after we eliminate the constants of integration to establish a posteriori the democracy of all paths between the given events. This approach affords a purely classical means of explicit evaluation of the propagator. In Sec. 2 we exhibit this elementary method to construct the propagators for free particle, harmonic oscillator and constant external field, where $V(x)$ is, at most, a second degree polynomial in x .

Elementary Procedure to Calculate the Propagator: Here we find the propagator for some simple systems:

a). $V(x) = 0$,

For free particle we have that $\exp\left(\frac{i}{\hbar} \int_0^t \frac{p^2}{2m} dt\right) = \exp\left(\frac{i}{\hbar} \frac{p^2}{2m} t\right)$ because the linear

momentum $p = m \frac{x - x'}{t}$ is a constant, hence from (4):

$$K(x, x', t) = N \exp\left[\frac{im}{2\hbar t} (x - x')^2\right], \quad (5)$$

which satisfies the Schrödinger equation. The relation:

$$\delta(x - a) = \lim_{\sigma \rightarrow 0} \frac{1}{\sigma \sqrt{\pi}} \exp\left[-\frac{(x - a)^2}{\sigma^2}\right], \quad (6)$$

permits to obtain N in (5) to verify (3), therefore the propagator for free particle is given by [1, 9-17]:

$$K(x, x', t) = \sqrt{\frac{m}{2\pi i\hbar t}} \exp\left[-\frac{m}{2i\hbar t}(x - x')^2\right]. \quad (7)$$

b). $V(x) = kx$.

For this linear potential the classical equations of motion give $\dot{x} = c - \frac{k}{m}t, x = x' - \frac{k}{2m}t^2 + ct, c = \text{constant}$,

$$L = \frac{m}{2}\dot{x}^2 - kx, \text{ then from (4):}$$

$$K(x, x', t) = N \exp\left\{\frac{i}{\hbar}\left[\frac{m}{2t}(x - x')^2 - \frac{k}{2}(x + x')t - \frac{k^2}{24m}t^3\right]\right\}, \quad (8)$$

which verifies the Schrödinger equation. From (6) and (8) we determine the value $N = \sqrt{\frac{m}{2\pi i\hbar t}}$ to satisfy (3), thus

we reproduce the expression reported in the literature [9, 10, 15, 18] for the propagator under a constant external field; the relation (8) with $k = 0$ gives (7) for free particle.

$$\text{c). } V(x) = \frac{1}{2}m\omega^2x^2.$$

For the harmonic oscillator the classical solution is given by:

$$x(t) = A \cos(\omega t + \varphi) = x' \cos(\omega t) - A \sin \varphi \sin(\omega t), x' = A \cos \varphi, \quad (9)$$

then,

$$\int_0^t L dt = \int_0^t \left(\frac{m}{2}\dot{x}^2 - \frac{m}{2}\omega^2x^2\right) dt = -\frac{m\omega}{2}[xx' \sin(\omega t) + A \sin \varphi (x \cos(\omega t) - x')],$$

hence (4) implies:

$$K(x, x', t) = N \exp\left\{\frac{im\omega}{2\hbar \sin(\omega t)}\left[(x^2 + x'^2)\cos(\omega t) - 2xx'\right]\right\}, \quad (10)$$

because from (9) we have that $A \sin \varphi = \frac{[x' \cos(\omega t) - x]}{\sin(\omega t)}$.

This expression (10) satisfies the Schrödinger equation and it is easy to see that $N = \sqrt{\frac{m\omega}{2\pi i\hbar \sin(\omega t)}}$ to verify (3)

in according with (6), thus we have obtained the formula for the harmonic oscillator propagator [1, 9-12, 14-16, 19-24].

CONCLUSIONS

The Lagrangians here considered are quadratic in the velocities where $V(x)$ is, at most, a second degree polynomial [10, 12, 25, 26]. It is simple to check that the propagators (7), (8) and (10) satisfy the Trotter formula [12, 27, 28]:

$$K(x, x', t) = \left(\frac{m}{2\pi i\hbar t}\right)^{r/2} \exp\left\{\frac{i}{2\hbar}\left[\frac{m}{t}(x - x')^2 - (V(x) + V(x'))t\right]\right\}, \quad (11)$$

for $t \ll 1$ with $r = 1$. We find that in the three cases here studied the Fujiwara's result [9] is verified:

$$N = \sqrt{\frac{i}{2\pi\hbar} \frac{\partial^2}{\partial x \partial x'}} \int_0^t L dt. \quad (12)$$

REFERENCES

- Park, D., 1992. Introduction to the quantum theory, McGraw-Hill, New York Chap. 13.
- Lanczos, C., 1997. Linear differential operators, Dover, New York.
- Dirac, P., 1933. The Lagrangian in quantum mechanics, Phys. Zeits. der Sowjetunion, 3(1): 64-72.
- Dirac, P., 1935. The principles of quantum mechanics, Clarendon Press, Oxford.
- Feynman, R.P., 1948. Space-time approach to non-relativistic quantum mechanics, Rev. Mod. Phys., 20(2): 367-387.
- Mehra, J., 2002. Richard Phillips Feynman. 11 May 1918-15 Feb. 1988, Biogr. Mem. Fell. R. Soc., 48: 97-128.
- Feynman, R.P., A.R. Hibbs and D.F. Styer, 2010. Quantum mechanics and path integrals: Emended edition, Dover, New York.

8. Eckert, M., 2013. Arnold Sommerfeld. Science, Life and Turbulent Times 1868-1951, Springer, Berlin.
9. Beauregard, L., 1966. Propagators in nonrelativistic quantum mechanics, Am. J. Phys., 34(4): 324-332.
10. Schulman, L.S., 1981. Techniques and applications of path integration, Wiley, New York Chap. 6.
11. Holstein, B.R., 1992. Topics in advanced quantum mechanics, Addison-Wesley, New York.
12. Khandekar, D.C., S.V. Lawande and K.V. Bhagwat, 1993. Path-integral methods and their applications, World Scientific, Singapore.
13. Taylor, E.F., S. Vokos and J. O'Meara, 1998. Teaching Feynman's sum-over-paths quantum theory, Computers in Phys., 12(2): 190-199.
14. Chaos, F.U., L. Chaos-Cador and E. Ley-Koo, 2002. Free-particle and harmonic-oscillator propagators in two and three dimensions, Revista Ciencias Exatas e Naturais, 4(2): 145-149.
15. Bouch, G. and P. Bowers, 2007. Evaluating Feynman path integrals via the polynomial path family, Math. Phys. Electr. J., 13: 1-29.
16. Moshinsky, M. and E. Sadurni, A. Del Campo, 2007. Alternative method for determining the Feynman propagator of a non-relativistic quantum mechanical problem, SIGMA, 3: 110-121.
17. Storey, P. and C. Cohen-Tannoudji, 1994. The Feynman path integral approach to atomic interferometry. A Tutorial, J. Phys. II France, 4: 1999-2027.
18. Holstein, B.R., 1997. The linear potential propagator, Am. J. Phys., 65(5): 414-418.
19. Donoghue, J.F. and B.R. Holstein, 1988. The harmonic oscillator via functional techniques, Am. J. Phys., 56(3): 216-22.
20. Alieva, T., 1996. On the self-fractional Fourier functions, J. Phys. A: Math. Gen., 29(15): L377-L379.
21. Cohen, S., 1998. Path integral for the quantum harmonic oscillator using elementary methods, Am. J. Phys., 66(6): 537-540.
22. Holstein, B.R., 1998. The harmonic oscillator propagator, Am. J. Phys., 66(7): 583-589.
23. Barone, F., H. Boschi-Filho and C. Farina, 2003. Three methods for calculating the Feynman propagator, Am. J. Phys., 71(5): 483-491.
24. Moriconi, L., 2004. An elementary derivation of the harmonic oscillator propagator, Am. J. Phys., 72(9): 1258-1259.
25. Montroll, E., 1952. Markoff chains, Wiener integrals and quantum theory, Commun. Pure Appl. Math., 5(4): 415-453.
26. Gelfand, I.M. and A.M. Yaglom, 1960. Integration in functional spaces and its applications in quantum physics, J. Math. Phys., 1(1): 48-69.
27. Nelson, E., 194. Feynman integrals and the Schrödinger equation, J. Math. Phys., 5(3): 332-343.
28. Makri, N., 1991. Feynman path integration in quantum dynamics, Computer Phys. Comm., 63(1-3): 389-414.