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A Formula for $_3F_2$ (2)

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Abstract: We give an elementary proof of the expression for $_3F_2$ $(-2n-1,\ \alpha,\ \beta+2;\ 2\alpha+2,\ \beta;\ 2)$ obtained by Rakha-Awad-Rathie.

Key words: Hypergeometric functions · Rakha-Awad-Rathie's formula

INTRODUCTION

In [1] we find the result:

$$A \equiv {}_3F_2(-2n-1,\alpha,\beta+2;\ 2\alpha+2,\beta;2) = \frac{(\beta-2\alpha)(\frac{3}{2})_n}{\beta(1+\alpha)(\alpha+\frac{3}{2})_n}, \tag{1}$$

in terms of the Pochhammer [2]-Barnes [3, 4] symbol. Here we use several formulas from Wolfram and the values [5-7]:

$$ {}_2F_1(-2n-1,\lambda;2\lambda;2) = 0,\quad {}_2F_1(-2n,\lambda;2\lambda;2) = \frac{(\frac{1}{2})_n}{(\lambda+\frac{1}{2})_n}, \tag{2}$$

to give an elementary deduction of (1).

Rakha-Awad-Rathie's Expression for $_3F_2$ (2): In Wolfram are the following relations for hypergeometric functions [8]:

$$b\ {}_3F_2(a,a_2,a_3;b,b_2;z) - a\ {}_3F_2(a+1,a_2,a_3;b+1,b_2;z) + (a-b)\ {}_3F_2(a,a_2,a_3;b+1,b_2;z) = 0, \tag{3}$$

$$c(a-b)\ {}_3F_2(a,b,a_3;c,b_2;z) - a(c-b)\ {}_3F_2(a+1,b,a_3;c+1,b_2;z) + b(c-a)\ {}_3F_2(a,b+1,a_3;c+1,b_2;z) = 0, \tag{4}$$

$$(a-c+1)(b-c+1)z\ {}_2F_1(a,b;c;z) + (c-1)(c-2)(z-1)\ {}_2F_1(a,b;c-2;z) + (c-1)[(a+b-2c+3)z+c-2]\ {}_2F_1(a,b;c-1;z) = 0 \tag{5}$$

$$ {}_2F_1(-n,a;2a+1;2) = \frac{\Gamma(a+\frac{1}{2})}{2\sqrt{n}}\left[\frac{(1+(-1)^n)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(a+\frac{n+1}{2}\right)} + \frac{(1-(-1)^n\Gamma\left(\frac{n+2}{2}\right)}{\Gamma(a+\frac{n+2}{2})}\right], \tag{6}$$

which permit to exhibit an elementary proof of (1). In fact, if into (3) we employ $a=-2n-1$, $a_2=a$, $a_3=\beta+2$, $b=\beta$, $b_2=2\alpha+2$ and $z=2$, we obtain the expression:

$$\beta A = (2n+\beta+1)\ B - (2n+1)\ C, \tag{7}$$

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such that

$$B \equiv {}_3F_2(-2n-1, \alpha, \beta+2; \beta+1, 2\alpha+2; 2), \quad C \equiv {}_3F_2(-2n, \alpha, \beta+2; \beta+1, 2\alpha+2; 2). \tag{8}$$

Now we must determine the quantities B and C, then into (4) we introduce the values $a = -2n-1$, $b = \alpha$ $c = \beta + 1$, $a_3 = \beta + 2$, $b_2 = 2\alpha + 2$ and $z = 2$ to deduce the relation:

$$B = \frac{(2n+1)(\beta+1-\alpha)}{(2n+\alpha+1)(\beta+1)} \; {}_2F_1(-2n, \alpha; 2\alpha+2; 2), \tag{9}$$

where was applied the first formula (2) with $\lambda + \alpha + 1$

If into (5) we use $a = -2n$, $b = \alpha$, $c = 2\alpha + 2$ and $z = 2$ we obtain that:

$$\frac{\alpha+1}{2\alpha+1} \; {}_2F_1(-2n, \alpha; 2\alpha+2; 2) = {}_2F_1(-2n, \alpha; 2\alpha+1; 2) - \frac{\alpha}{2n+2\alpha+1} \; {}_2F_1(-2n, \alpha; 2\alpha; 2), \tag{10}$$

besides from (2) and (6) are immediate the values:

$${}_2F_1(-2n, \alpha; 2\alpha; 2) = {}_2F_1(-2n, \alpha; 2\alpha+1; 2) = \frac{(\frac{1}{2})n}{(\alpha+\frac{1}{2})_n}, \tag{11}$$

hence from (9) and (10):

$${}_2F_1(-2n, \alpha; 2\alpha+2; 2) = \frac{(2n+\alpha+1)(\frac{3}{2})_n}{(2n+1)(\alpha+1)(\alpha+\frac{3}{2})_n}, \quad B = \frac{(\beta+1-\alpha)(\frac{3}{2})_n}{(\beta+1)(\alpha+1)(\alpha+\frac{3}{2})_n}. \tag{12}$$

To calculate C, into (4) we employ $a = -2n$, $b + \alpha$, $c = \beta + 1$, $a_3 = \beta + 2$, $b_2 = 2\alpha + 2$ and $z = 2$, thus:

$$C = \frac{2}{(2n+\alpha)(\beta+1)}[n(\beta+1-\alpha) \; {}_2F_1(-2n+1, \alpha; 2\alpha+2; 2) + \frac{\alpha}{2}(2n+\beta+1) \; {}_2F_1(-2n, \; \alpha+1; 2\alpha+2; 2)] \tag{13}$$

and from (2) with $\lambda = \alpha + 1$:

$${}_2F_1(-2n, \alpha+1; 2\alpha+2; 2) = \frac{(\frac{1}{2})_n}{(\alpha+\frac{3}{2})_n}, \tag{14}$$

besides from (5) with $a = -2n+1$, $b = \alpha$, $c = 2\alpha + 2$ and $z = 2$:

$${}_2F_1(-2n+1, \alpha; 2\alpha+2; 2) = \frac{2\alpha+1}{\alpha+1} \; {}_2F_1(-2n+1, \alpha; 2\alpha+1; 2) \overset{(6)}{=} \frac{(2n+2\alpha+1)(\frac{3}{2})_n}{(2n+1)(\alpha+1)(\alpha+\frac{3}{2})_n}. \tag{15}$$

Now we put (14) and (15) into (13) to obtain:

$$C = \frac{(2n+2\alpha+1)(\frac{3}{2})_n}{(2n+1)(2n+\alpha)(\beta+1)(\alpha+\frac{3}{2})_n}[\frac{\alpha(2n+\beta+1)}{2n+2\alpha+1} + \frac{2n(\beta+1-\alpha)}{\alpha+1}], \tag{16}$$

Finally, the application of (12) and (16) in (7) permits to construct the formula (1) deduced by Rakha-Awad-Rathie [1], q.e.d.

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