Computational and Applied Mathematical Sciences 1 (1): 14-17, 2016
ISSN 2222-1328

DOI: 10.5829/idosi.cams.2016.14.17

(Antisymmetric Matrix)$_{3x3}$

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Abstract: We determine the exponential function of certain antisymmetric matrix, which represents an arbitrary 3-rotation. Besides, we exhibit the relationship of this result with the generators of the groups SO(3) and SU(2).

Key words: Exponential function ・ 3-Rotations ・ SU(2) and SO(3)

INTRODUCTION

For the antisymmetric matrix:

$$F = \begin{pmatrix} 0 & -l_3 & l_2 \\ l_3 & 0 & -l_1 \\ -l_2 & l_1 & 0 \end{pmatrix}, \quad l_1^2 + l_2^2 + l_3^2 = 1, \tag{1}$$

we calculate the exponential function $e^{F\tau}$ to obtain the matrix representation of an arbitrary 3-rotation and the corresponding generators of the groups SU(2) and SO(3), in harmony with the results of Ryder [1].

Matrix Exponential Function and SO(3): From (1) it is immediate to deduce the relations:

$$F^{2n} = -(-1)^n F^2, n = 2,3,..., \quad F^{2m+1} = (-1)^m F, \ m = 1,2,... \tag{2}$$

where:

$$F^2 = \begin{pmatrix} l_1^2 - 1 & l_1 l_2 & l_1 l_3 \\ l_2 l_1 & l_2^2 - 1 & l_2 l_3 \\ l_3 l_1 & l_3 l_2 & l_3^2 - 1 \end{pmatrix}, \quad (F^2)_{jk} = l_j l_k - \delta_{jk}, \tag{3}$$

hence:

$$\exp(F\tau) = I + \left(\tau - \frac{\tau^3}{3!} + \frac{\tau^5}{5!} - ...\right)F + \left(\frac{\tau^2}{2!} - \frac{\tau^4}{4!} + \frac{\tau^6}{6!} - ...\right)F^2, \quad \Gamma = 1 - \cos\tau,$$

$$= I + \sin + \tau F + \Gamma F^2 = \begin{pmatrix} l_1^2\Gamma + \cos\tau & l_1 l_2 \Gamma - l_3 \sin\tau & l_1 l_3 \Gamma + l_2 \sin\tau \\ l_1 l_2 \Gamma + l_3 \sin\tau & l_2^2 \Gamma + \cos\tau & l_1 l_3 \Gamma - l_1 \sin\tau \\ l_1 l_3 \Gamma - l_2 \sin\tau & l_2 l_3 \Gamma + l_1 \sin\tau & l_3^2 \Gamma + \cos\tau \end{pmatrix} = R_{\hat{l}}(\tau), \tag{4}$$

which is the Euler's expression [2, 3] for a rotation by an angle τ around of the axis $\hat{l} = (l_1, l_2, l_3)$,

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The matrix F accepts the splitting:

$$F = -il_1\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & -i\\ 0 & i & 0\end{pmatrix} - il_2\begin{pmatrix}0 & 0 & i\\ 0 & 0 & 0\\ -i & 0 & 0\end{pmatrix} - il_3\begin{pmatrix}0 & -i & 0\\ i & 0 & 0\\ 0 & 0 & 0\end{pmatrix} = -i\vec{J}.\hat{l}, \tag{5}$$

where J_x, J_y,J_z are Hermitian generators of SO(3), thus (4) coincides with (2.35) of [1]:

$$R_{\hat{l}}(\tau) = \exp\left[-i\vec{J}.\hat{l}\tau\right], \quad \det R_{\hat{l}}(\tau) = 1, \tag{6}$$

that is:

$$[-i\frac{\partial}{\partial_\tau}R_{\hat{l}}(\tau)]_{r=0} = \begin{cases} J_{x,} & \hat{l} = (-1,0,0),\\ J_{y,} & \hat{l} = (0,-1,0),\\ J_{z,} & \hat{l} = (0,0-1).\end{cases} \tag{7}$$

If we introduce the Euler-Olinde Rodrigues parameters [4]:

$$a_0 = \cos\left(\frac{\tau}{2}\right), \quad a_k = l_k \sin\left(\frac{\tau}{2}\right), k = 1,2,3, \tag{8}$$

then $a_0^2 + a_1^2 + a_2^2 + a_3^2 = 1$ and (4) takes the form:

$$R_{\hat{l}}(\tau) = \begin{pmatrix} 1-2(a_2^2+a_3^2) & 2(a_1a_2 - a_3a_0) & 2(a_1a_3 + a_2a_0)\\ 2(a_1a_2 + a_3a_0) & 1-2(a_1^2+a_3^2) & 2(a_2a_3 - a_1a_0)\\ 2(a_1a_3 - a_2a_0 & 2(a_1a_0 + a_2a_3) & 1-2(a_1^2+a_2^2)\end{pmatrix}, \tag{9}$$

which establishes a connection with unitary real quaternions [5].

In terms of the Cayley-Klein's rotation parameters [6]:

$$\alpha = a_0 - ia_3, \quad \beta = -a_2 - ia_1, \quad \alpha\overline{\alpha} + \beta\overline{\beta} = 1, \tag{10}$$

the matrix (9) adopts the structure (2.54) of Ryder [1]:

$$R_{\hat{l}}(\tau) = \begin{pmatrix} \frac{1}{2}(\alpha^2 + \overline{\alpha}^2 - \beta^2 - \overline{\beta}^2) & -\frac{i}{2}(\alpha^2 + \overline{\alpha}^2 + \beta^2 - \overline{\beta}^2) & -\alpha\beta - \overline{\alpha}\overline{\beta}\\ \frac{i}{2}(\alpha^2 - \overline{a}^2 - \beta^2 + \overline{\beta}^2) & \frac{1}{2}(\alpha^2 + \overline{a}^2 - \beta^2 + \overline{\beta}^2) & i(-\alpha\beta + \overline{\alpha}\overline{\beta})\\ \alpha\overline{\beta} + \overline{\alpha}\beta & i(\overline{\alpha}\beta + \alpha\overline{\beta}) & a\overline{\alpha} - \beta\overline{\beta} \end{pmatrix}. \tag{11}$$

SU(2) and 3-Rotations: The matrix (4) permits rotate the position vector $\vec{r}$:

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = R_{\hat{l}}(\tau) \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad R_{\hat{l}} R_{\hat{l}}^T = I, \tag{12}$$

which implies the interesting vectorial expression of Euler [2, 3, 7]:

$$\vec{r}\,' = (1-\cos\tau)\left(\hat{l}.\vec{r}\right)\hat{l} + \cos\tau\ \vec{r} + \sin\tau\ \hat{l}\ x\ \vec{r}. \tag{13}$$

The relation (12) is equivalent to the Olinde Rodrigues-Cartan's formula [8-11]:

$$\begin{pmatrix} z' & x'-iy' \\ x'+iy' & -z' \end{pmatrix} = U \begin{pmatrix} z & x-iy \\ x+iy & -z \end{pmatrix} U^\dagger, \quad \det U = 1, \quad UU^\dagger = 1, \tag{14}$$

with the unitary matrix [12]:

$$U = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} = \begin{pmatrix} e^{\frac{i\varphi}{2}} & 0 \\ 0 & e^{\frac{i\varphi}{2}} \end{pmatrix} \begin{pmatrix} \cos(\frac{\theta}{2}) & i\sin(\frac{\theta}{2}) \\ i\sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \end{pmatrix} \begin{pmatrix} e^{\frac{i\psi}{2}} & 0 \\ 0 & e^{\frac{i\psi}{2}} \end{pmatrix}, \tag{15}$$

where φ, θ, ψ are the Euler's angles [2, 13]:

$$\alpha = \cos\left(\frac{\theta}{2}\right) \exp\left[\frac{1}{2}(\varphi+\psi)\right], \quad \beta = i\sin\left(\frac{\theta}{2}\right) \exp\left[\frac{i}{2}(\varphi-\psi)\right]. \tag{16}$$

From (8) and (10):

$$\alpha = \cos\left(\frac{\tau}{2}\right) - il_3 \sin(\frac{\tau}{2}), \quad \beta = -(l_2 + il_1)\sin\left(\frac{\tau}{2}\right), \tag{17}$$

then U takes the form (2.61) of Ryder [1]:

$$U = I\cos\left(\frac{\tau}{2}\right) - i\vec{\sigma}.\hat{l}\sin\left(\frac{\tau}{2}\right) = \exp\left[-i\frac{\vec{\sigma}}{2}.\hat{l}\tau\right], \tag{18}$$

with the Pauli's matrices [14]:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{19}$$

hence, in analogy with (7), the Hermitian generators of SU(2) are given by:

$$[-i\frac{\partial}{\partial\tau}U]_{\tau=0} = \begin{cases} \frac{\sigma_x}{2}, & \hat{l} = (-1,0,0), \\ \frac{\sigma_y}{2}, & \hat{l} = (0,-1,0), \\ \frac{\sigma_z}{2}, & \hat{l} = (0,0,-1). \end{cases} \tag{20}$$

Thus, we see that the exponential function of the antisymmetric matrix (1) gives an excellent platform to study 3-rotations and the corresponding generators of SU(2) and SO(3).

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