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Construction and Reconstruction of Dissemilattices

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Abstract: In 1967, J. Plonka introduced a construction called "Plonka sum" of algebras of the same type indexed by a set with a semilattice structure. Examples of algebras that can be represented as Plonka sum are provided by dissemilattice. The aim of this paper is to present construction methods of dissemilattice that on the one hand allow to compose a new dissemilattice from a family of dissemilattice indexed by a set with a dissemilattice structure and on the other hand to reconstruct a dissemilattice from any quotient of it and its fibers. In this way the described methods of compositions of dissemilattices provides a very useful tool for describing a structure of the greater class of dissemilattices.

Key wrods: Semilattice · Dissemilattice · Joiu-semilattice · Stemmered Semilattice

INTRODUCTION

Perliminaries: A semilattice (S,.) is a commutative idempotent semigroup. Such a structure yields a partial order $\leq$ on the set S on setting $x \leq y$ iff $x.y = x$. A semilattice with such a partial order is called a meet-semilattice. Similarly, a semilattice (S,+), one may consider the partial order $\leq_+$ on the set S on setting $x \leq_+ y$ iff $x + y = y$. A semilattice with such a partial order is called a join-semilattice.

A set S with two semilattice operations + and. is called a bisemilattice. If (S,+,.) satisfy in addition to the obsorptive law; i.e. for $x,y = \in S$, $x + x.y = x$ the distributive law; i.e. $x,y,z \in S$, $x.(y + z) = x,y + x.z$ then (S,+,.) is called a meet- distributive bisemilattice, i.e. (S,+,.) is a dissemilattice.

A stammered semilattice is a bisemilattice which further satisfies the identity $x + y = x.\ y$. In this case, the set S is ordered by the relation $\leq_+ = \leq$.

The Basic Construction:

Definition 2.1: Let (I,+,.) be a dissemilattice, for each $i \in I$, let (B_i.+,.) be a family of dissemilattices be given. For each pair (i, j) of elements of I such that $I \leq_+ j$ or $j \leq i$, let a mapping $\varphi_{i,j}$: $B_i \rightarrow B_j$ be given such that the following are satisfied for $i, j, k \in I$, $x_i \in B_i$, $y_j \in B_j$:

- $\varphi_{i.i} = 1B_{i,j}$;
- $(x_i\varphi_{i.i+j} + y_j\varphi_{j.i+j})\varphi_{i+j,k} = x_i\varphi_{i.k} + y_j\varphi_{j.k}$, for $i+j \leq_+ k$;
- $(x_i\varphi_{i.ij} \cdot y_j\varphi_{j.ij})\varphi_{ij,k} = x_i\varphi_{i.k} \cdot y_j\varphi_{j.k}$, for $k \leq ij$

Then the disjoint union $B = \bigcup_{i\in I} B_i$, $i \in I$ with two binary operations defined on B by:

$$x_i + y_j := x_i\varphi_{i.i+j} + y_j\varphi_{j.i+j}$$
$$x_i \cdot y_j := x_i\varphi_{i.i.j} \cdot y_j\varphi_{j.i.j}$$

is called sum of dissemilattices (B_i, +,.) over the dissemilattice (I,+,.) by $\varphi_{i,j}$.

Note that, for $i \leq_+ k$, the mapping $\varphi_{i,k}$ is a join-semilattice homomorphism and for $k \leq i$, the mapping $\varphi_{i,k}$ is a meetsemilattice homomorphism. To prove this, put i=j in 2 & 3 we have,

$$(x_i + y_j)\varphi_{i,k} = x_i\varphi_{i.k} + y_j\varphi_{j.k};$$
$$(x_i \cdot y_j)\varphi_{i,k} = x_i\varphi_{i.k} \cdot y_j\varphi_{j.k}.$$

Example 2.2: Let (I, +,.) be a dissemilattice, with $I = \{1,2,3\}$, such that $1 \leq 2$, $1 \leq 3$, $2.3 = 1$ and $1 \leq_+ 2 \leq_+ 3$ For each $i \in I$, let $B_1 = \{c, d\}$, $B_2 = \{a\}$ and $B_3 = \{b\}$ such that $c \leq d$ and $c \leq_+ d$ be a family of dissemilattices. Further, define a mapping $\varphi_{i,j} : B_i \rightarrow B_j$ by the following figures.

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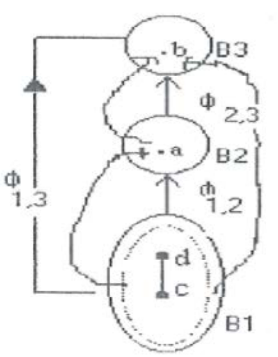
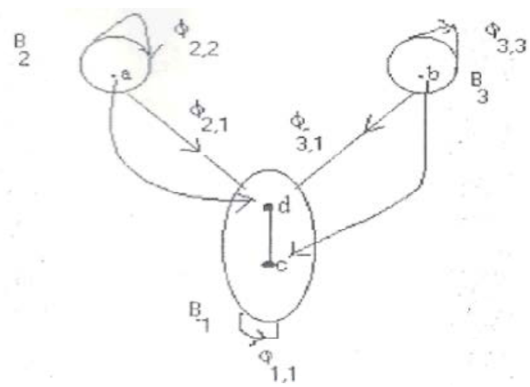




Then the disjoint union $B = \dot{\bigcup_{i\in I}} B_i$, $i\in I$ equipped with the two binary operations defined on B by:

$$x_i + y_j = x_i\varphi_{i.i+j} + y_j\varphi_{j.i+j}$$
$$x_i . y_j = x_i\varphi_{i.i.j} . y_j\varphi_{j.i.j}$$

is the sum of the dissemilattices $(B_i,+,.)$ over the dissemilattice $(I,+,.)$ by $\varphi_{i,j}$.

Theorem 2.3: The sum $(B,+,.)$ of dissemilattice $(B_i,+,.)$, $i\in I$ over a dissemilattice $(I,+,.)$ is a dissemilattice.

Proof: Clearly both operations + and • are idempotent and commutative. Associativity follows from 2 & 3, i.e. let $x_i \in B_i$, $y_j \in B_j$, $z_k \in B_k$, then;

$$(x_i + y_j) + z_k = (x_i + y_j)\varphi_{i+j,i+j+k} + z_k\varphi_{k.i+j+k}$$
$$= (x_i\varphi_{i,i+j+k} + y_j\varphi_{j,i+j+k}) + z_k\varphi_{k.i+j+k}$$
$$= x_i\varphi_{i,i+j+k} + (y_j\varphi_{j,i+j+k} + z_k\varphi_{k.i+j+k})$$
$$= x_i + (y_j + z_k)$$

It follows that $(B,+)$ is a join semilattice. Similarly, $(B,.)$ is a meet semilattice.

Now, let $x_i \in B_i$, $y_j \in B_j$, $z_k \in B_k$, then;

$$x_i . (y_j + z_k) = x_i\varphi_{i,ij+ik} . (y_j + z_k)\varphi_{j+k,ij+ik}$$
$$= x_i\varphi_{i,ij+ik} . (y_j\varphi_{j,ij+ik} + z_k\,\varphi_{k,ij+ik})$$
$$= x_i\varphi_{i,ij+ik} . y_j\varphi_{j,ij+ik} + x_i\varphi_{i,ij+ik} . z_k\,\varphi_{k,ij+ik})$$
$$= (x_i . y_j)\,\varphi_{ij,ij+ik} + (x_i . z_k)\,\varphi_{ik,ij+ik}$$
$$= (x_i\,\varphi_{i,ij} . y_j\,\varphi_{j,ij}) + (x_i\varphi_{i,ik} . z_k\varphi_{k,ik}) = (x_i . y_j) + (x_i . z_k)$$

Thus, The distributive law hold and hence $(B,+,.)$ is a dissemilattice.?

Sum of Dissemilattice over a Stammered Semilattice:

Definition 2.5: Let Let $(I,+,.)$ be a stammered semilattice, for each $i\in I$, let $(B_i,+,.)$ be a family of dissemilattice be given. For each pair (i,j) with $j \le i$, define a homomorphism $\varphi_{i,j}$: $B_i \to B_j$ be given such that the following are satisfied for $i, j, k \in I$, $x_i \in B_i$, $y_j \in B_j$:

- $\varphi_{i.i} = 1_{B_i\,j}$;
- $(x_i + y_j)\varphi_{ij,k} = x_i\varphi_{i.k} + y_j\varphi_{j.k}$, for $ij \ge k$;
- $(x_i . y_j)\varphi_{ij,k} = x_i\varphi_{i.k} . y_j\varphi_{j.k}$, for $ij \ge k$.

Then the disjoint union $B = \dot{\bigcup_{i\in I}} B_i$, $i \in I$ with two binary operations defined on B by:

$$x_i + y_j := x_i\varphi_{i.ij} + y_j\varphi_{j.ij}, \text{ for } ij \ge k;$$
$$x_i . y_j := x_i\varphi_{i.i.j} . y_j\varphi_{j.i.j}, \text{ for } ij \ge k$$

is called sum of dissemilattices $(B_i,+,.)$ over the stammered semilattice $(I,+,.)$ by $\varphi_{i,j}$.

Theorem 2.6: The sum $(B,+,.)$ of dissemilattice $(B_i,+,.)$, $i\in I$ over a stammered semilattice $(I,+,.)$ is a dissemilattice.

Proof: Clearly both operations + and • are idempotent and commutative. Associativity follows from 2 & 3, i.e. let $x_i \in B_i$, $y_j \in B_j$, $z_k \in B_k$, then

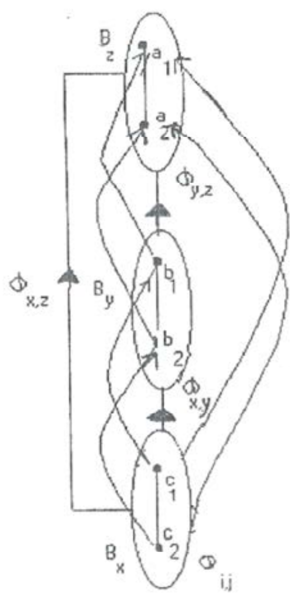
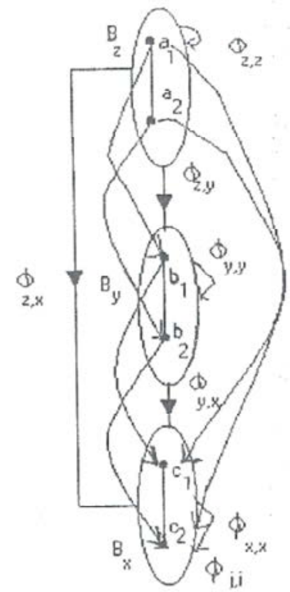
$$(x_i + y_j) + z_k = (x_i + y_j)\varphi_{i+j,i+j+k} + z_k\varphi_{k.i+j+k}$$
$$= (x_i\varphi_{i,i+j+k} + y_j\varphi_{j,i+j+k}) + z_k\varphi_{k.i+j+k}$$
$$= x_i\varphi_{i,i+j+k} + (y_j\varphi_{j,i+j+k} + z_k\varphi_{k.i+j+k})$$
$$= x_i + (y_j + z_k)$$

It follows that $(B,+)$ is a join semilattice. Similarly, $(B,.)$ is a meet semilattice.
Now, let $x_i \in B_i$, $y_j \in B_j$, $z_k \in B_k$, then

$x_i.(y_j + z_k) = x_i\varphi_{i,ij+ik}.(y_j + z_k)\varphi_{j+k,ij+ik}$
$= x_i\varphi_{i,ij+ik}.(y_j\varphi_{j,ij+ik} + z_k\ \varphi_{k,ij+ik})$
$= x_i\varphi_{i,ij+ik}.y_j\varphi_{j,ij+ik} + x_i\varphi_{i,ij+ik}.z_k\ \varphi_{k,ij+ik})$
$= (x_i.y_j)\ \varphi_{ij,ij+ik} + (x_i.z_k)\ \varphi_{ik,ij+ik}$
$= (x_i\ \varphi_{i,ij}.y_j\ \varphi_{j,ij}) + (x_i\varphi_{i,ik}.z_k\varphi_{k,ik}) = (x_i\ .y_j) + (x_i.z_k).$

Thus, The distributive law hold and hence $(B,+,.)$ is a dissemilattice.?

Example 2.7: Let $(I,+,.)$ be a stammered semilattice, with $I = \{x, y, z\}$, such that $1 \leq 2 \leq 3$. For each $i \in I$, let $B_x, = \{a, b\}$, $B_y = \{c, d\}$ and $B_z = \{e, f\}$ such that $a \leq b$ and $c \leq d$ and $e \leq f$ be a family of dissemilattice. Further, for each pair $(i, j) \in I \times I$ define a mapping $\varphi_{i,j} : B_i \rightarrow B_j$ by the following figures.

Then the disjoint union $B = \dot{\bigcup_{i \in I}} B_i$, $i \in I$ equipped with the two binary operations defined on B by:

$$x_i + y_j = x_i\varphi_{i.ij} + y_j\varphi_{j.ij}$$
$$x_i.y_j = x_i\varphi_{i.ij}.y_j\varphi_{j.ij}$$

is the sum of the dissemilattices $(B_i,+,.)$ over the stammered semilattice $(I,.,.)$.

Theorem 2.8: The sum $(B,+,.)$ of stammered semilattices $(B_i,+,.)$, $i \in I$ over a stammered semilattice $(I,+,.)$ is a stammered semilattice.

Proof: Let $x_i, y_j \in B$. It suffices to show that $x_i + y_j = x_i.y_j$.

We have $x_i + y_j, = x_i\varphi_{i,ij} + y_j\varphi_{j,ij}$ since $x_i\varphi_{i,ij}$, $y_j\varphi_{j,ij}$ are in B_{ij} and $(B_{ij},+,.)$ is a stammered semilattice, then $x\,\varphi_{\ i,ij} + y_j\varphi_{j,ij} = x_i\varphi j_{,ij}.y_j\varphi_{j,ij}$ but $x_i\varphi_{i,ij}.y_j\varphi_{j,ij} = x_i.y_j$.

It follows that $(B,+,.)$ is a stammered semilattice. ?

Definition 2.9: Let $(I,+,.)$ be the stammered semilattice $(I,.,.)$. For each $i \geq j \in I$, let $(B_i, +,.)$ be an algebra and $\varphi_{i,j}$: $B_i \rightarrow B_j$ be a homomorphism such that

- $\varphi_{i,i} = 1_{Bi,j}$;
- $x_i\varphi_{i,j}\varphi_{j,k} = x_i\varphi_{i,k}$, for $i \geq j \geq k$;

then the disjoint union $B = \dot{\bigcup_{i \in I}} B_i$, $i \in j$ equipped with the two binary operations defined on B by:

$$x_i + y_j = x_i\varphi_{i,ij} + y_j\varphi_{j,ij}$$
$$x_i.y_j = x_i\varphi_{i,i.j}.y_j\varphi_{j,i.j}$$

is called the plonka sum of the algebras $(B_i,+,.)$ over the stammered semilattice $(I,.,.)$.

Note that if all $\varphi_{i,j}$ are dissemilattice homomorphism and take the condition of the functoriality $x_i,\varphi_{i,j},\varphi_{j,k} = x_i\varphi_{i,k}$, for $i \geq j \geq k$ instead of the conditions 2 & 3 in Definition 2.4, then the definition becomes the Plonka sum of dissemilattice.

Theorem 2.10: If $(I,.,.)$ is a stammered semilattice and all $(B_i, +,.)$ are lattices, then the sum $(B,+,.)$ of $(B_i,+,.)$ over $(I,.,.)$ is a quasilattice, i.e. a bisemilattice satisfies the identities, for $x, y, z \in B$

- $x + y = x \Rightarrow xz + yz = xz$
- $x.y = x \Rightarrow (x + z).(y + z) = x + z.$

Proof: It can be easily verified that both $(B,+)$ and $(B,.)$ are semilattice, i.e. $(B,+,.)$ is a bisemlattice. Now let $x_i + y_j = x_i$, then;

$$
\begin{aligned}
& x_i . z_k + y_j . z_k = (x_i\varphi_{i,ik} . z_k\varphi_{k,ik}) + (y_j\varphi_{j,jk} . z_k\varphi_{k,jk}) \\
&= (x_i\varphi_{i,ik} . z_k\varphi_{k,ik})\varphi_{ik,ijk} + (y_j\varphi_{j,jk} . z_k\varphi_{k,jk})\varphi_{jk,ijk} \\
&= (x_i\varphi_{i,ijk} . z_k\varphi_{k,ijk}) + (y_j\varphi_{j,ijk} . z_k\varphi_{k,ijk}) \\
&= (z_k\varphi_{k,ijk} . x_i\varphi_{i,ijk}) + (z_k\varphi_{k,ijk} . y_j\varphi_{j,ijk}) \\
&= z_k\varphi_{k,ijk} . (x_i\varphi_{i,ijk} + y_j\varphi_{j,ijk}) = (x_i\varphi_{i,ij} + y_j\varphi_{j,ij})\varphi_{ij,ijk} . z_k\varphi_{k,ijk} \\
&= (x_i\varphi_{i,ij} + y_j\varphi_{j,ij}). z_k = x_i . z_k
\end{aligned}
$$

Similarly, $x.y = x \Rightarrow (x + z).(y + z) = x + z$.?

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