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Homotopy Analysis Method for Solving the Equation Governing the Unsteady Flow of a Polytropic Gas

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Abstract: In this paper, an application of Homotopy Analysis Method (HAM) is applied to solve the equation governing the unsteady flow of a polytropic gas. Comparison are made between the Adomian decomposition method and homotopy analysis method. The results reveal that the homotopy analysis method is very effective and simp le and gives the exact solution.

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INTRODUCTION

In 1992, Liao [1] employed the basic ideas of the homotopy in topology to propose a general analytic method for linear and nonlinear problems, namely Homotopy Analysis Method (HAM) [2-4]. Based on homotopy of topology, the validity of the HAM is independent of whether or not there exist small prameters in the considered equation. Therefore, the HAM can overcome the foregoing restrictions and limitations of perturbation techniques. This method has been successfully applied to solve many types of nonlinear problems. In this paper we prove theorem of Convergence of homotopy analysis method and apply this method to solve the equation governing the unsteady flow of a polytropic gas.

Basic idea of HAM: We consider the following differential equation

$$N[u(\tau)] = 0 \tag{1}$$

where N is a nonlinear operator, τ denotes independent variable, $u(\tau)$ is an unknown function, respectively. function, respectively. For simplicity, we ignore all boundary or initial conditions, which can be treated in the similar way. By means of generalizing the traditional homotopy method, Liao [5] construct the so-called zero-order deformation equation

$$(1-p)L[\phi(r,t;p) - u_0(r,t)] = p\hbar H(r,t)N[\phi(r,t;p)] \tag{2}$$

where $p\in[0,1]$ is the embedding parameter, $\hbar \neq 0$ is a nonzero auxiliary parameter, $H(\tau)\neq 0$ is an auxiliary function, $u_0(\tau)$ is an initial guess of $u(\tau)$, $\phi(r, p)$ is a unknown function and L is an auxiliary linear operator with the property

$$L[f(\tau)] = 0 \text{ where } f(r) = 0 \tag{3}$$

It is important, that one has great freedom to choose auxiliary things in HAM. Obviously, when p = 0 and p = 1, it holds

$$\phi(\tau;0) = u_0(\tau), \qquad \phi(\tau;1) = u(\tau) \tag{4}$$

respectively. Thus, as p increases from 0 to 1, the solution $\phi(r; p)$ varies from the initial gusse $u_0(\tau)$ to the solution $u(\tau)$. Expanding $\phi(r; p)$ in Taylor series with respect to p, we have

$$\phi(\tau;p) = u_0(\tau) + \sum_{m=1}^{+\infty} u_m(\tau)p^m \tag{5}$$

where

$$u_m(\tau) = \frac{1}{m!}\frac{\partial^m \phi(\tau;p)}{\partial p^m}\Big|_{p=0} \tag{6}$$

If the auxiliary linear operator, the auxiliary parameter h and the auxiliary function are so properly chosen, the series (5) convergent at p =1, then we have

$$u(\tau) = u_0(\tau) + \sum_{m=1}^{+\infty} u_m(\tau) \tag{7}$$

which must be one of solutions of original nonlinear equation, as proved by [5]. As h =-1 and $H(\tau) = 1$ Eq. (2) becomes

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$$(1-p)L[\phi(\tau,p)-u_0(\tau)]+pN[\phi(\tau,p)]=0 \qquad (8)$$

which is used mostly in the homotopy perturbation method [6], where as the solution obtained directly, without using Taylor series [7]. According to the definition (6), the governing equation can be deduced from the zero-order deformation equation (2). Define the vector

$$\vec{u}_n=\{u_0,u_1,\ldots,u_n\}$$

Differentiating equation (2) m times with respect to the embedding parameter p and finally dividing them by m!, we have the so-called m-th order deformation equation

$$L[u_m-\chi_m u_{m-1}]=hH(\tau)R_m(\vec{u}_{m-1}) \qquad (9)$$

where

$$R_m(\vec{u}_{m-1})=\frac{1}{(m-1)!}\frac{\partial^{m-1}N[\phi(\tau;p)]}{\partial p^{m-1}}\Big|_{p=0} \qquad (10)$$

and

$$\chi m=\begin{cases}0, & m\le 1\\ 1, & m>1.\end{cases} \qquad (11)$$

It should be emphasized that $u_m(\tau)$ for $m\ge 1$ is governed by the linear equation (9) under the linear boundary condition that come from original problem, which can be easily solved by symbolic computation software such as Matlab. If equation (1) admist unique solution, then this method will produse the unique solution. If equation (1) dos not possess unique solution, the HAM will give a solution among many other (possible) solution.

CONVERGENCE OF HAM

In this section, we will prove that, as long as the solution series (7) given by the homotopy analysis method is convergent, it must be the solution of the considered nonlinear problem.

Theorem 2.1: As long as the series

$$u_0(t)+\sum_{m=1}^{+\infty}u_m(t)$$

is convergent, where um(t) is governed by the high-order deformation equation (9) under the definitions (10) and (11), it must be a solution of Equation (1). proof. Let

$$s(t)=u_0(t)+\sum_{m=1}^{+\infty}u_m(t)$$

denote the convergent series. Using (9) and (11), we have

$$\begin{aligned}hH(t)\sum_{m=1}^{+\infty}R_m(\vec{u}_{m-1})&=\sum_{m=1}^{+\infty}L\left[u_m(t)-\chi_m u_{m-1}(t)\right]\\&=L\left[\sum_{m=1}^{+\infty}u_m(t)-\sum_{m=1}^{+\infty}\chi_m u_{m-1(t)}\right]\\&=L\left[(1-\chi_2)\sum_{m=1}^{+\infty}u_m(t)\right]\\&=L\left[(1-\chi_2)(s(t)-u_0(t))\right]\end{aligned}$$

which gives, since $h\neq 0$ and $H(r)\neq 0$ from (3),

$$\sum_{m=1}^{+\infty}R_m(\vec{u}_{m-1})=0 \qquad (12)$$

On the other side, we have according to the definition (10), that

$$\sum_{m=1}^{+\infty}R_m(\vec{u}_{m-1})=\sum_{m=1}^{+\infty}\frac{1}{(m-1)!}\frac{\partial^{m-1}N[\phi(t;q)]}{\partial q^{m-1}}\Big|_{q=0} \qquad (13)$$

In general, $\phi(r;p)$ does not satisfy the original nonlinear equation (1). Let

$$\varepsilon(t;q)=N[\phi(t;q)]$$

denote the residual error of Equation (1). Clearly,

$$\varepsilon(t;q)=0$$

Corresponds to the exact solution of the original equation (1). According to the above definition, the Maclaurin series of the residual error $\phi(t;p)$ about the embedding parameter q is

$$\sum_{m=0}^{+\infty}\frac{1}{m!}\frac{\partial^m\varepsilon(t;q)}{\partial q^m}\Big|_{q=0}=\sum_{m=0}^{+\infty}\frac{1}{m!}\frac{\partial^m N[\phi(t;q)]}{\partial q^m}\Big|_{q=0}$$

When q = 1, the above expression gives, using (13)

$$\varepsilon(t;q)=\sum\frac{1}{m!}\frac{\partial^m\varepsilon(t;q)}{\partial q^m}\Big|_{q=0}=0$$

This means, according to the definition of $\phi(t;p)$, that we gain the exact solution of the original equation (1) when q. Thus, as long as the series

$$u_0(t)+\sum_{m=1}^{+\infty}u_m(t)$$

is convergent, it must be one solution of the original equation (1). This ends the proof.

APPLICATION

In order to assess the advantages and the accuracy of homotopy analysis method for solving nonlinear systems, we will consider the following example.

Example: The equation governing the unsteady ow of a polytropic gas in two dimensions is given by [8-10] as follows

$$\begin{aligned} &u_t+uu_x+vu_y+\frac{p_x}{\rho}=0\\ &v_t+uv_x+vv_y+\frac{p_y}{\rho}=0\\ &\rho_t+u\rho_x+v\rho_y+\rho(u_x+v_y)=0\\ &p_t+up_x+vp_y+\gamma(u_x+v_y)=0 \end{aligned} \tag{14}$$

where ρ is the density, p the pressure, u and v the velocity components in the x and y directions, respectively and the adiabatic index γ is the ratio of the specific heats. With the initial data:

$$\begin{aligned} &u(x,y,0)=e^{x+y}\\ &v(x,y,0)=-1-e^{x+y}\\ &\rho(x,y,0)=e^{x+y}\\ &p(x,y,0)=c \end{aligned} \tag{15}$$

Note that the selection of equations (15) that are obtained from [8] the fluid is incompressible and invisid (no viscose). The exact solution of equations (15) is

$$\begin{aligned} &u(x,y,0)=e^{x+y+t}\\ &v(x,y,0)=-1-e^{x+y+t}\\ &\rho(x,y,0)=e^{x+y+t}\\ &p(x,y,0)=c \end{aligned} \tag{16}$$

To solve the Eqs. (15) by means of homotopy analysis method, we choose the linear oprator

$$L_i[\phi(x,y,t;p)]=\frac{\partial\phi(x,y,t;p)}{\partial t},\ i=1,2,3,4 \tag{17}$$

with the property

$$L_i[c]=0,\ i=1,2,3,4$$

where c are integral constant. The inverse operator L^{-1} is given by

$$L_i^{-1}(\cdot)=\int_0^t(\cdot)dt,\ i=1,2,3,4 \tag{18}$$

Now we define a nonlinear operators as

$$\begin{aligned} N_1&=\frac{\partial\phi_1}{\partial t}+\phi_1\frac{\partial\phi_1}{\partial x}+\phi_2\frac{\partial\phi_1}{\partial y}+\frac{\frac{\partial\phi_3}{\partial x}}{\phi_4}\\ N_2&=\frac{\partial\phi_2}{\partial t}+\phi_1\frac{\partial\phi_2}{\partial x}+\phi_2\frac{\partial\phi_2}{\partial y}+\frac{\frac{\partial\phi_3}{\partial y}}{\phi_4}\\ N_3&=\frac{\partial\phi_4}{\partial t}+\phi_1\frac{\partial\phi_4}{\partial x}+\phi_2\frac{\partial\phi_4}{\partial y}+\phi_4\left(\frac{\partial\phi_1}{\partial x}+\frac{\partial\phi_2}{\partial y}\right)\\ N_4&=\frac{\partial\phi_3}{\partial t}+\phi_1\frac{\partial\phi_3}{\partial x}+\phi_2\frac{\partial\phi_3}{\partial y}+\gamma\phi_3\left(\frac{\partial\phi_1}{\partial x}+\frac{\partial\phi_2}{\partial y}\right) \end{aligned} \tag{19}$$

Thus, by the above definitions we obtain the mth-order deformation equations

$$\begin{aligned} &L_1[u_m(x,y,t)-\chi_m u_{m-1}(x,y,t)]=h_1H_1R_{1,m}\\ &L_2[v_m(x,y,t)-\chi_m v_{m-1}(x,y,t)]=h_2H_2R_{2,m}\\ &L_3[\rho_m(x,y,t)-\chi_m\rho_{m-1}(x,y,t)]=h_3H_3R_{3,m}\\ &L_4[p_m(x,y,t)-\chi_m p_{m-}1(x,y,t)]=h_4H_4R_{4,m} \end{aligned} \tag{20}$$

where

$$H_i=H_i(x,y,t),R_{i,m}=R_{i,m}(\vec{u}_{m-1},\vec{v}_{m-1},\vec{\rho}_{m-1},\vec{p}_{m-1})$$

for i = 1,2,3,4 and

$$\begin{aligned} R_{1,m}&=\frac{\partial u_{m-1}}{\partial t}+\sum_{i=0}^{m-1}\left(u_i\frac{\partial u_{m-1-i}}{\partial x}+v_i\frac{\partial u_{m-1-i}}{\partial y}\right)+A_1(\rho,p_x)\\ R_{2,m}&=\frac{\partial v_{m-1}}{\partial t}+\sum_{i=0}^{m-1}\left(u_i\frac{\partial v_{m-1-i}}{\partial x}+v_i\frac{\partial v_{m-1-i}}{\partial y}\right)+A_2(\rho,p_y)\\ R_{3,m}&=\frac{\partial\rho_{m-1}}{\partial t}+\sum_{i=0}^{m-1}\left(u_i\frac{\partial\rho_{m-1-i}}{\partial x}+v_i\frac{\partial\rho_{m-1-i}}{\partial y}+\rho_i\frac{\partial u_{m-1-i}}{\partial x}\right.\\ &\left.+\rho_i\frac{\partial v_{m-1-i}}{\partial y}\right)\\ R_{4,m}&=\frac{\partial p_{m-1}}{\partial t}+\sum_{i=0}^{i=m-1}\left(u_i\frac{\partial p_{m-1-i}}{\partial x}+v_i\frac{\partial p_{m-1-i}}{\partial y}+\gamma P_i\frac{\partial u_{m-1-i}}{\partial x}\right.\\ &\left.+\gamma p_i\frac{\partial v_{m-1-i}}{\partial y}\right) \end{aligned}$$

where

$$A_1(\rho,p_x)=\frac{p_{0x}}{\rho_0}+\frac{\rho_0p_{1,x}-\rho_1p_{0,y}}{\rho_0^2}$$

$$A_2(\rho,p_y)=\frac{p_{0y}}{\rho_0}+\frac{\rho_0p_{1,y}-\rho_1p_{0,y}}{\rho_0^2}$$

Now the solutions of the mth-order deformation equations (20)

$$\begin{aligned} u_m(x,y,t) &= \chi_m u_{m-1}(x,y,t) + H_1 h_1 L_1^{-1}[R_{1,m}] \\ v_m(x,y,t) &= \chi_m v_{m-1}(x,y,t) + H_2 h_2 L_2^{-1}[R_{2,m}] \\ \rho_m(x,y,t) &= \chi_m \rho_{m-1}(x,y,t) + H_3 h_3 L_3^{-1}[R_{3,m}] \\ p_m(x,y,t) &= \chi_m p_{m-1}(x,y,t) + H_4 h_4 L_4^{-1}[R_{4,m}] \end{aligned} \tag{21}$$

We start with an initial approximations

$$\begin{aligned} u_0(x,y,t) &= e^{x+y} \\ v_0(x,y,t) &= -1 - e^{x+y} \\ \rho_0(x,y,t) &= e^{x+y} \\ p_0(x,y,t) &= c \end{aligned}$$

and by means of the above iteration formula (20) if

$$H_i(x,y,t) = 1, h_i = -1$$

for i = 1,2,3,4 we can obtain directly the other components as

$$\begin{aligned} &u_1 = \frac{t}{1!}e^{x+y},\ \ v_1 = -\frac{t}{1!}e^{x+y},\ \rho_1 = \frac{t}{1!}e^{x+y},\ \ p_1 = 0 \\ &u_2 = \frac{t^2}{2!}e^{x+y},\ v_2 = -\frac{t^2}{2!}e^{x+y},\ \rho_2 = \frac{t^2}{2!}e^{x+y}, p_2 = 0 \\ &u_3 = \frac{t^3}{3!}e^{x+y},\ v_3 = -\frac{t^3}{3!}e^{x+y},\ \rho_3 = \frac{t^3}{3!}e^{x+y},\ p_3 = 0 \\ &\vdots \end{aligned}$$

Continuing the expansion to the last term gives the solution of(13) as

$$\begin{aligned} u(x,y,t) &= \sum_{n=0}^{+\infty} \frac{t^n}{n!} e^{x+y} = e^{x+y+t} \\ v(x,y,t) &= -1 - \sum_{n=0}^{+\infty} \frac{t^n}{n!} e^{x+y} = -1 - e^{x+y+t} \\ \rho(x,y,t) &= \sum_{n=0}^{+\infty} \frac{t^n}{n!} e^{x+y} = e^{x+y+t} = e^{x+y+t} \\ p(x,y,t) &= c \end{aligned}$$

which is exactly the same as obtained by Adomian decomposition method [11].

CONCLUSION

In this paper, the HAM was used to obtain the exact solutions of the Equation Governing the Unsteady Flow of a Polytropic Gas. The comparison between the HAM and ADM was made and it was found that HAM is more effiective than ADM. Hence, it may be concluded that this method is a powerful and an efficient technique in finding the exact solutions for wide classes of problems. Furthermore, the advantage of this method is the fast convergence of the solutions by means of the auxiliary parameter h and the freedom of choosing h for HAM gives us more accuracy than ADM. It is also worth mentioning to this end that for the example considered, we have shown that ADM are special case of HAM. The computations associated with the example in this Letter were performed using Matlab 7.

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