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A Short Note on Entropy Ordering Property for Concomitants of Order Statistics

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Abstract: Let (X_i, Y_i) $i = 1,2,..., n$ be a random sample of size n from a continuous bivariate distribution. If the pairs are ordered by their X values, then the Y values associated with the r-th order statistic $X_{(r)}$ of X will be denoted by $Y_{[r]}$, $1 \le r \le n$ and be called the concomitant of the r-th order statistic. In this paper, we present a very short note on entropy ordering for concomitants of order statistics.

Key words: Concomitants · Entropy ordering · Order statistics

INTRODUCTION

Let $(X_1, Y_1), (X_2, Y_2),..., (X_n, Y_n)$ be a random sample of size n from a continuous bivariate distribution. If we arrange the X's in ascending order as $X_{(1)} \le X_{(2)} \le ... \le X_{(n)}$, then the Y's associated with these order statistics are denoted by $Y_{[1]}, Y_{[2]}, ... , Y_{[n]}$ and are called concomitants of order statistics. An excellent review of work on concomitants of order statistics is available in David and Nagaraja [1]. In the following Section, we present some results that relate entropy ordering property for concomitants of order statistics to other well-known ordering of random variables.

Main Results: In this Section, first we briefly review the various notions of stochastic ordering among concomitants of order statistics.So we need the following definitions in which X and Y denote random variables with distributions $F_x(x)$ and $G_y(y)$, density functions $f_x(x)$ and $g_Y(y)$ and survival functions $\overline{F}_X(x) = 1 - F_x(x)$ and $\overline{G}_Y(y) = 1 - G_Y(y)$.

Definition 2.1: *X is said to be smaller thanY according to*

(a) Stochastic ordering (denoted by $X \overset{st}{\le} Y$) if $\overline{F}_X(x) \le \overline{G}_Y(x)$ for all x.
(b) Entropy ordering(denoted by $X \overset{e}{\le} Y$) if $H(X) \le H(Y)$.
(c) Dispersion ordering (denoted by $X \overset{disp}{\le} Y$) if

$$F_x^{-1}(\beta) - F_x^{-1}(\alpha) \le G_Y^{-1}(\beta) - G_Y^{-1}(\alpha)$$
for all $0 < a \le \beta < 1$.

Definition 2.2: *We say that Y is stochastically increasing (decreasing) in X (denoted by SI(Y|X)) (SD(Y|X)) if* $P(Y > y|X = x)$ *is increasing(decreasing) function in x for all y.*

Definition 2.3: *A random variable X is said to have a decreasing (an increasing) failure rate (DFR (IFR)) if its failure rate function* $\lambda_x(t) = \dfrac{f_x(t)}{1 - F_x(t)}$ *is decreasing (increasing) in* $t > 0$.

Khaledi and Kochar [3] obtained some results of stochastically comparing the concomitant $Y_{[r]}$'s under different kinds of dependence between X and Y. They proved that if Y is stochastically increasing (decreasing) in X, then the concomitant variables $Y_{[r]}$'s are stochastically increasing (decreasing). This result is shown by the following expressions.

$$(i)\ SI(Y|X) \Rightarrow Y_{[r]} \overset{st}{\le} Y_{[k]} \quad for \quad 1 \le r < k \le n,$$
$$(ii)\ SD(Y|X) \Rightarrow Y_{[r]} \overset{st}{\ge} Y_{[k]} \quad for \quad 1 \le r < k \le n, \qquad (1)$$

They also proved that if the conditional hazard rate of Y given $X = x$ $(\lambda(y|x))$ is decreasing function in x and y, then the concomitants have DFR distributions and are ordered according to dispersive ordering.

Theorem 2.1: *Suppose that the conditional hazard rate of Y given* $X = x$ $(\lambda(y|x))$ *is decreasing function in x and y, then for* $1 \le r < k \le n$,

$$(i)\ Y_{[k]} \overset{e}{\le} Y_{[r]} \ \ if\ P(Y > y|X = x)\ is\ increasing\ function\ in\ x\ for\ all\ y,$$
$$(ii)\ Y_{[r]} \overset{e}{\le} Y_{[k]} \ \ if\ p(Y > y|X = x)\ is\ increasing\ function\ in\ x\ for\ all\ y \qquad (2)$$

The inequalities in (2) are reversed in the case that $\lambda(y|x)$ is increasing in x and y.

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Proof. (i): Using (1), we have

$$SD\,(Y|X) \Rightarrow Y_{[k]} \overset{st}{\leq} Y_{[r]}\, for \;\; 1 \leq r < k \leq n,$$

Also, since $\lambda(y|x)$ is decreasing function in y for each fixed x, we conclude that $Y_{[r]}$ is DFR for $1 \leq r \leq n$. Thus,

$$Y_{[k]} \overset{st}{\leq} Y_{[r]} \Leftrightarrow E_{g[k]}\left[log\vartheta_{[r]}(y)\right] \geq E\vartheta_{[r]}\left[log\,\vartheta_{[r]}(y)\right], \quad (3)$$

Now, the discrimination information between $Y_{[k]}$ and $Y_{[r]}$ is

$$K(\vartheta_{[k]}:\vartheta_{[r]}) = \int_{-\infty}^{+\infty}\vartheta_{[k]}(y)\log(\frac{\vartheta_{[k]}(y)}{\vartheta_{[r]}(y)})dy$$
$$= -H(Y_{[k]}) - E_{\vartheta[k]}\left[\log\vartheta_{[r]}(y)\right] \geq 0, \quad (4)$$

By Eqs.(3) and (4) the proof is complete.

(ii): The proof is similar to that of part (i), by using the fact that $Y_{[r]} \overset{st}{\leq} Y_{[k]}$.

It is well known that $Y_{[r]} \overset{disp}{\leq} Y_{[k]}$ implies $Y_{[r]} \overset{st}{\leq} Y_{[k]}$ and obviously $Y_{[r]} \overset{disp}{\leq} Y_{[k]}$ implies that $Y_{[r]} \overset{e}{\leq} Y_{[k]}$ for $1 \leq r < k \leq n$.

Corollary 2.1: *Let $Y_{[r]}$ be a concomitant of order statistics having a DFR (IFR) distribution. If $Y_{[r]} \overset{st}{\leq} Y_{[k]}$, then $Y_{[r]} \overset{e}{\leq} Y_{[k]}\left(Y_{[k]} \overset{e}{\leq} Y_{[r]}\right)$* for all $1 \leq r < k \leq n$.

The following example gives an application of theorem 2.1.

Example 2.1: Let (X_i, Y_i), $i = 1,2, \ldots, n$ be a random sample from Gumbel's *bivariate exponential distribution (see Johnson and Kotz [2], p.261) with density*

$$f(x,y) = \exp(-x - y - \theta xy)[(1+\theta x)(1+\theta y) - \theta],\; 0 \leq \theta \leq 1, x, y > 0. \quad (5)$$

In this case, the conditional hazard rate of Y given $X = x$ is

$$\lambda\,(y\,|\,x) = \frac{(1+\theta x)(1+\theta y) - \theta}{1+\theta y}, \quad (6)$$

Which increasing in y and x. Since the conditional survival function

$$P(Y > y\,|\,X = x) = \exp[-y\;(1+\theta x)](1+\theta y)$$

is decreasing function in x, It follows from theorem 2.1 that $Y_{[r]}$ has IFR distribution and $Y_{[r]} \overset{e}{\leq} Y_{[k]}$ for $1 \leq r < k \leq n$.

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