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Solution of the Lane-Emden Type Equations Arising in Astrophysics Using Differential Transformation Method

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Abstract: In this paper we propose differential transformation method (DTM) for solving some well-known classes of Lane-Emden type equations. They are categorized as singular initial value problems. The principle of differential transformation is briefly introduced and is then applied in the derivation of a set of difference equations for the some well-known classes of Lane-Emden type equations. The solutions are subsequently solved by a process of inverse transformation. To illustrate the reliability of the method, some special cases of the equations are solved as test examples. The new method reduces the solution of a problem to the solution of a system of algebraic equations. Differential transformation method has prefect properties that make them useful to achieve this goal. We obtained the exact solution of these examples by DTM.

Key words: Differential transformation method · Lane-Emden type equations · Spherical cloud of gas · Astrophysics

INTRODUCTION

Singular initial value problems in the second order ordinary differential equations occur in several models of mathematical physics and astrophysics [1-3] such as the theory of stellar structure, the thermal behavior of a spherical cloud of gas, isothermal gas spheres and theory of thermionic currents which are modeled by means of the following Lane-Emden equation:

$$u''(x)+\frac{\alpha}{x}u'(x)+f(x,u)=g(x), 0<x\le 1, \alpha\ge 0 \quad (1)$$

Under the following initial conditions

$$u(0)=A, u'(0) \quad (2)$$

Where:
A and B are constants, $f(x, u)$ is a continuous real valued function and $g(x)\in C[0,1]$.

Lane-Emden type equations are categorized as singular initial value problems. These equations describe the temperature variation of a spherical gas cloud under the mutual attraction of its molecules and subject to the laws of classical thermodynamics. The polytropic theory of stars essentially follows out of thermodynamic considerations that deal with the issue of energy transport, through the transfer of material between different levels of the star. These equations are one of the basic equations in the theory of stellar structure and have been the focus of many studies [4-20].

In this paper, we extend the application of the differential transformation method [21], which is based on Taylor series expansion, to construct analytical approximate solutions of the initial value problem (1)-(2). The concept of differential transformation was introduced first by Zhou [21] and it was applied to solve linear and nonlinear initial value problems in electric circuit analysis. With this technique, it is possible to obtain highly accurate results or exact solutions for differential equations.

This Paper Is Arranged as Follows: In Section 2 we survey several methods that have been used to solve Lane-Emden type equations. In Section 3, the principle of differential transformation is briefly introduced. In Section 4 the proposed method is applied to some types of Lane-Emden equations and exact solutions are obtained. Finally we give a brief conclusion in the last section.

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Methods Have Been Proposed to Solve Lane-emden Type Equations: Recently, many analytical methods have been used to solve Lane-Emden type equations, the main difficulty arises in the singularity of the equations at $x = 0$. Currently, most techniques which were used in handling the Lane-Emden type problems are based on either series solutions or perturbation techniques. Bender *et al.* [4] proposed a new perturbation technique based on an artificial parameter δ, the method is often called δ-method. Mandelzweig and Tabakin [5] used the quasilinearization approach to solve the standard Lane-Emden equation. This method approximates the solution of a nonlinear differential equation by treating the nonlinear terms as a perturbation about the linear ones and unlike perturbation theories is not based on the existence of some small parameters. Shawagfeh [6] applied a nonperturbative approximate analytical solution for the Lane-Emden type equation using the Adomian decomposition method. His solution was in the form of a power series. He used Padé approximants method [22, 23] to accelerate the convergence of the power series.

In [7], Wazwaz employed the Adomian decomposition method [24, 25] with an alternate framework designed to overcome the difficulty of the singular point. It was applied to the differential equations of Lane-Emden type. Further author of [8] used the modified decomposition method for solving the analytical treatment of nonlinear differential equations such as the Lane-Emden type equation. Liao [26] provided an analytical algorithm for Lane Emden type equations. This algorithm logically contains the well-known Adomian decomposition method. Different from all other analytical techniques, this algorithm itself provides us with a convenient way to adjust convergence regions even without Padé technique. J.H. He [10] employed Ritz's method to obtain an analytical solution of the problem. By the semi-inverse method, a variational principle is obtained for the Lane-Emden type equation. Parand *et al.* [27-29] presented two numerical techniques to solve higher ordinary differential equations such as Lane-Emden. Their approach was based on the rational Chebyshev and rational Legendre Tau methods. Ramos [11-14] solved Lane-Emden equations through different methods. Author of [12] presented the linearization method for singular initial-value problems in second-order ordinary differential equations such as Lane-Emden. These methods result in linear constant-coefficients ordinary differential equations which can be integrated analytically, thus yielding piecewise analytical solutions and globally smooth solutions. Later this author [14] developed piecewise-adaptive decomposition methods for the solution of nonlinear ordinary differential equations. In [13], series solutions of the Lane-Emden type equation have been obtained by writing this equation as a Volterra integral equation and assuming that the nonlinearities are sufficiently differentiable. These series solutions have been obtained by either working with the original differential equation or transforming it into an ordinary differential equation that does not contain the first-order derivatives. Series solutions to the Lane-Emden type equation have also been obtained by working directly on the original differential equation or transforming it into a simpler one. Yousefi [15] presented a numerical method for solving the Lane-Emden equations. He converted Lane-Emden equations to integral equations, using integral operator and then he applied Legendre wavelet approximations. Bataineh *et al.* [16] presented an algorithm based on homotopy analysis method (HAM) [30] to obtain the exact and/or approximate analytical solutions of the singular IVPs of the Emden-Fowler type equation. In [17], Chowdhury and Hashim presented an algorithm based on the homotopy-perturbation method (HPM) [31-33] to solve singular IVPs of time-independent equations. Aslanov [18] introduced a further development in the Adomian decomposition method to overcome the difficulty at the singular point of non-homogeneous, linear and nonlinear Lane-Emden-like equations. Dehghan and Shakeri [19] applied an exponential transformation to the Lane Emden type equations to overcome the difficulty of a singular point at $x = 0$ and solved the resulting nonsingular problem by the variational iteration method [34, 35]. Yildirim and Özis [36] presented approximate solutions of a class of Lane-Emden type singular IVPs problems, by the variational iteration method. Marzban *et al.* [37] used a method based upon hybrid function approximations. They used the properties of hybrid of block pulse functions and Lagrange interpolating polynomials together for solving the nonlinear second-order initial value problems and the Lane-Emden type equation. Recently, Singh *et al.* [38] provided an efficient analytic algorithm for Lane-Emden type equations using modified homotopy analysis method; also they used some well-known Lane-Emden type equations as test examples. We refer the interested reader to [39, 40] for analysis of the Lane-Emden type equation based on the Lie symmetry approach.

Fundamentals of Differential Transformation Method: Let $u(x)$ be analytic in a domain D and let $x = x_i$ represent any point in D. The function $u(x)$ is then represented by one power series whose center is located at x_i. The Taylor series expansion function of $u(x)$ is in form of:

$$u(x)=\sum_{k=0}^{\infty}\frac{(x-x_i)^k}{k!}\left[\frac{d^k u(x)}{dx^k}\right]_{x=x_i}\quad \forall x\in D \tag{3}$$

The particular case of Eq. (3) when $x = x_i$ is referred to as the Maclaurin series of $u(x)$ and is expressed as:

$$u(x)=\sum_{k=0}^{\infty}\frac{(x)^k}{k!}\left[\frac{d^k u(x)}{dx^k}\right]_{x=0}\quad \forall x\in D \tag{4}$$

As explained in [41] the differential transformation of the function $u(x)$ is defined as follows:

$$U(k)=\sum_{k=0}^{\infty}\frac{(H)^k}{k!}\left[\frac{d^k u(x)}{dx^k}\right]_{x=0} \tag{5}$$

Where:
$u(x)$ is the original function and $U(k)$ is the transformed function. The differential spectrum of $U(k)$ is confined within the interval $x \in [0,H]$, where H is a constant. The differential inverse transform of $U(k)$ is defined as follows:

$$u(x)=\sum_{k=0}^{\infty}\left(\frac{x}{H}\right)^k U(k) \tag{6}$$

It is clear that the concept of differential transformation is based upon the Taylor series expansion. The values of function $U(k)$ at values of argument k are referred to as discrete, i.e.$U(0)$ is known as the zero discrete,$U(1)$ as the first discrete, etc. the more discrete available, the more precise it is possible to restore the unknown function. The function $u(x)$ consists of T-function $U(k)$ and its value is given by the sum of the T-function with $\left(\frac{x}{H}\right)^k$ as its coefficient. In real applications, at the right choice of constant H, the larger values of argument k the discrete of spectrum reduce rapidly. The function $u(x)$ is expressed by a finite series and Eq. (6) can be written as:

$$u(x)=\sum_{k=0}^{n}\left(\frac{x}{H}\right)^k U(k) \tag{7}$$

Mathematical operations performed by differential transform method are listed in Table 1.

Table 1: The fundamental operations of differential transform method

Original function	Transformed function
$u(x) = \alpha f(x)\pm\beta g(x)$	$U(k) = \alpha F(k)\pm\beta G(k)$
$u(x)=\frac{df(x)}{dx}$	$U(k) = (k + 1)F(k + 1)$
$u(x)=\frac{d^2 f(x)}{dx^2}$	$U(k) = (k + 1)(k + 2)F(k + 2)$
$u(x) = x^m$	$U(k)=\delta(k-m)=\begin{cases}1 & k=m\\ 0 & k\neq m\end{cases}$
$u(x) = \exp(x)$	$U(k)=\frac{1}{k!}$
$u(x) = f(x)g(x)$	$U(k)=\sum_{l=0}^{k}F(l)G(k-l)$

Applications: In this section we apply differential transformation method (DTM) for the computation of Lane-Emden type equations.

Example 1: *(The standard Lane-Emden equation)*

For $f(x, u) = u(x)$, $g(x) = 0$, $\alpha=2$, $A = 1$ and $B = 0$, Eq. (1) is the standard Lane-Emden equation that was used to model the thermal behavior of a spherical cloud of gas acting under the mutual attraction of its molecules and subject to the classical laws of thermodynamics [6, 42].

$$u''(x)+\frac{2}{x}u'(x)+u(x)=0,x\geq 0, \tag{8}$$

Subject to the initial conditions

$$u(0) = 1, u'(0) = 0 \tag{9}$$

By multiplying both sides of Eq. (8) by x and then taking differential transformation of both sides of the resulting equation using Table 1 and the following recurrence relation is obtained:

$$U(k+1)=\frac{1}{(k+1)(k+2)}\times\left(-\sum_{l=0}^{k}\delta(l-1)U(k-l)\right) \tag{10}$$

By using Eqs. (5) and (9), the following transformed initial conditions for H=1 can be obtained:

$$U(0) = 1 \tag{11}$$

$$U(1) = 0 \tag{12}$$

Substituting Eqs. (11) And (12) at $k = 1, 2, 3$, into Eq. (10), we have

$$U(2) = -\frac{1}{6}, U(3) = 0, U(4) = \frac{1}{120}, U(5) = 0, U(6) = -\frac{1}{5040}, U(7) = 0, U(8) = \frac{1}{362880}, \dots \tag{13}$$

Using Eqs. (11-13) and the inverse transformation rule in Eq. (7), we get the following solution:

$$u(x) = 1 - \frac{1}{6}x^2 + \frac{1}{120}x^4 - \frac{1}{5040}x^6 + \frac{1}{362880}x^8 - \cdots \tag{14}$$

Or

$$u(x) = \frac{\sin(x)}{x} \tag{15}$$

Example 2: For $f(x, u) = u(x)$, $g(x) = 6+12x+x^2+x^3$, $\alpha=2$, $A = 0$ and $B = 0$, Eq. (1) will be one of the Lane-Emden type equations that is.

$$u''(x) + \frac{2}{x}u'(x) + u(x) = 6 + 12x + x^2 + x^3, x \geq 0, \tag{16}$$

Subject to the initial conditions

$$u(0) = 0, u'(0) = 0 \tag{17}$$

By multiplying both sides of Eq. (8) by x and then taking differential transformation of both sides of the resulting equation using Table 1 and the following recurrence relation is obtained:

$$U(k+1) = \frac{1}{(k+1)(k+2)} \times \begin{pmatrix} 6\delta(k-1) + 12\delta(k-2) + \delta(k-3) + \\ \delta(k-4) - \sum_{l=0}^{k} \delta(l-1)U(k-l) \end{pmatrix} \tag{18}$$

By using Eqs. (5) and (17), the following transformed initial conditions for H=1 can be obtained:

$$U(0) = 0 \tag{19}$$

$$U(1) = 0 \tag{20}$$

Substituting Eqs. (19) And (20) at $k = 1, 2, 3$, into Eq. (18), we have

$$U(2) = 1, U(3) = 1, U(4) = 0, U(5) = 0, U(6) = 0, U(7) = 0, U(8) = 0, \dots \tag{21}$$

Using Eqs. (19-21) and the inverse transformation rule in Eq. (7), we get the following solution:

$$u(x) = x^2 + x^3 \tag{22}$$

This equation has been solved by [12, 36, 43, 45] with linearization, VIM, HPM and TSADM methods respectively. We applied the DTM to solve Eq. (16).

Example 3: For $f(x, u) = xu(x)$, $g(x) = x^5 - x^4 + 44x^2 - 30x$, $\alpha=8$, $A = 0$ and $B = 0$, Eq. (1) will be one of the Lane-Emden type equations that is absorbing to solve

$$u''(x) + \frac{8}{x}u'(x) + xu(x) = x^5 - x^4 + 44x^2 - 30x, x \geq 0, \tag{23}$$

Subject to the initial conditions

$$u(0) = 0, u'(1) = 0 \tag{24}$$

By multiplying both sides of Eq. (23) by x and then taking differential transformation of both sides of the resulting equation using Table 1 and the following recurrence relation is obtained:

$$U(k+1) = \frac{1}{(k+1)(k+8)} \times \begin{pmatrix} \delta(k-6) - \delta(k-5) + 44\delta(k-3) - \\ 30\delta(k-2) - \sum_{l=0}^{k} \delta(l-2)U(k-l) \end{pmatrix} \tag{25}$$

By using Eqs. (5) and (24), the following transformed initial conditions for H=1 can be obtained:

$$U(0) = 0 \tag{26}$$

$$U(1) = 0 \tag{27}$$

Substituting Eqs. (26) And (27) at $k = 1, 2, 3$, into Eq. (25), we have

$$U(2) = 0, U(3) = -1, U(4) = 1, U(5) = 0, U(6) = 0, U(7) = 0, U(8) = 0, \dots \tag{28}$$

Using Eqs. (26-28) and the inverse transformation rule in Eq. (7), we get the following solution:

$$u(x) = -x^3 + x^4 \tag{29}$$

This type of equation has been solved by [12, 17, 44, 45] with linearization, HPM, HAM and TSADM methods respectively. We applied the DTM to solve this type equation (23).

Example 4: For $f(x, u) = -2(2x^2+3)\,u(x)$, $g(x) = 0$, $\alpha = 2$, $A = 1$ and $B = 0$, Eq. (1) will be one of the Lane-Emden type equations that is

$$u''(x) + \frac{2}{x}u'(x) - 2\left(2x^2 + 3\right)u(x) = 0, x \geq 0, \tag{30}$$

Subject to the initial conditions

$$u(0) = 1,\ u'(1) = 0 \tag{31}$$

By multiplying both sides of Eq. (30) by x and then taking differential transformation of both sides of the resulting equation using Table 1 and the following recurrence relation is obtained:

$$U(k+1) = \frac{1}{(k+1)(k+2)} \times \begin{pmatrix} \delta(l-3)U(k-l) + \\ 4\sum_{l=0}^{k} 6\sum_{l=0}^{k} \delta(l-1)U(k-l) \end{pmatrix} \tag{32}$$

By using Eqs. (5) and (31), the following transformed initial conditions for $H=1$ can be obtained:

$$U(0) = 0 \tag{33}$$

$$U(1) = 0 \tag{34}$$

Substituting Eqs. (33) And (34) at $k = 1,2,3,\ldots$ into Eq. (32), we have

$$U(2) = 1, U(3) = 0, U(4) = \frac{1}{2}, U(5) = 0, U(6) = \frac{1}{6}, U(7) = 0, U(8) = \frac{1}{24}, \ldots \tag{35}$$

Using Eqs. (33-35) and the inverse transformation rule in Eq. (7), we get the following solution:

$$u(x) = 1 + x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6 + \frac{1}{24}x^8 + \frac{1}{120}x^{10} + \ldots \tag{36}$$

Or

$$u(x) = e^{x^2} \tag{37}$$

This type of equation has been solved by [12, 36, 43] with linearization, VIM and HPM methods respectively. We applied the DTM method to solve Eq. (30).

CONCLUSION

In this study, the differential transformation method is implemented to the Lane-Emden differential equations as singular initial value problems. Four equations are solved and exact solutions are obtained. It is shown that differential transformation method is a very fast convergent, precise and cost efficient tool for solving the Lane-Emden equations.

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