World Applied Sciences Journal 9 (10): 1155-1157, 2010
ISSN 1818-4952

Modified Iteration Methods to Solve System *Ax* = *b*

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Abstract: A new method for solving linear systems, $Ax = b$, is derived. It can be considered as a modification of the coefficient matrix, A, by fuzzy determinat of fuzzy memebership matrix and then apply Jacobi or Gauss-Seidel iteration methods or any iteration methods.

Key words: Fuzzy membership matrix · Jacobi method · Gauss-Seidel method · Max-Min algebra · Linear systems · Iteration method · Hungarian algorithm

INTRODUCTION

Consider a system of n equations

$$Ax = b, \quad (1)$$

Where:
$A \in R^{n\times n}$ and $b,x \in R^n$ with A, b are known and x is unknown. The coefficient matrix A is split into

$$A = M - N$$

Where:
M is nonsingular. Then a linear stationary iterative method for solving (1) can be described as follows:

$$x^{k+1} = Tx^k - M^{-1}b \; k = 0,1,2,..., \quad (2)$$

Where:
$T = M^{-1} N$ is the iteration matrix. It is well known that for nonsingular systems the iterative method (2) is convergent if and only if

$$\rho(T) = \max\{|\mu|, \mu \in \sigma(T)\}<1.$$

for more detail see [1-7]. We will transform the original system (1) into the preconditioned form $PAx = Pb$.

First we give some definitions:

Definition 1.1 *[8]Matrix A called fuzzy memebership matrix if $\alpha_{ij} \in [0,1]$ for all index i,j and Fuzzy (membership) determinant of a square matrix A is defined by*

$$Fdet(A) = \max_{(h_1 h_2 ... h_n)} \{\min\{\alpha_{1h_1}, \alpha_{2h_2}, ..., \alpha_{nh_n}\}\},$$

Where:
max is for all permutations $(h_1h_2...h_n)$ of indices $\{1,2,...,n\}$. Then we can extend this definition into real square matrix A:

Definition 1.2 *For a real square matrix A, fdet(A) can be define by*

$$fdet(A) = \max_{(h_1 h_2 \cdots h_n)} [\min\{|a_{1h_1}|,|a_{2h_2}|,...,|a_{nh_n}|\}],$$

Where:
max is for all permutations $(h_1h_2...h_n)$ of indices $\{1,2,...,n\}$.

In the next section we will introduce algorithms, first to calculate $fdet(A)$ and second principle algorithm to how commute rows of matrix A or as matrix PA and in last section will given numerical results.

Algorithms: Now, we are ready to state an algorithm for a given matrix $A \in R^{n\times n}$ for evaluating the determinant defined on previous section.

Algorithm 1 *"To calculate fdet(A)"*.
STEP 1: Start with $A \in R^{n\times n}$ and set $N = M = \{1,2,...,n\}$.
STEP 2: Set $M_{iR} := \max_{j\in N} |\alpha_{ij}|$ and $M_{jC} := \max_{i\in N} |\alpha i_j|$ fonr $i,j \in N$ and so $fd := \min\{\min_k\{M_{kR}\}, \min_k\{M_{kC}\}\}$.
STEP 3: Reduce A to $\breve{A}$ by $[\breve{A}]_{ij} := \min\{|[A]_{ij}|, fd\}$.
STEP 4: If fd is occurred in every rows and columns of $\breve{A}$ as a permutation, then $fdet(A) := fd$ and stop.

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STEP 5: (Else) Choose maximum element of $\breve{A}$ which is smaller than *fd*, say $[\breve{A}]_{pq}$ and set $fd := [\breve{A}]_{pq}$ and $[\breve{A}]_{ij} := min\{|[\breve{A}]_{ij}|, fd\}$ and return to step 3.

Also we can use the method Hungarian algorithm to find *fdet(A)*.

Now we can used Algorithm 1 and then commute rows of A to find the best iteration matrix, say PA, from Jacobi method or Gauss-Seidel method or any iteration methods.

Algorithm 2: *"Principal algorithm"*.

Step 1: Start with $A \in R^{n\times n}$ are known and set $N = M = \{1,2,...,n\}$.

Step 2: Using algorithm 1 and recognized indexes i, σ_i which.

$$fdet(A) = |a_{i\sigma_i}| = \min_j\{|a_{j\sigma_j}|\}$$

$$fdet(A) = \max_{\sigma\in F} min\{|a_{k\sigma_k}|\}$$

Where:

F is all one to one functions from M to N. If index i is not unique, then choose index i_0 that *fdet* matrix B, arising row i_0 and column σ_{i_0} from matrix A be maximum.

Step 3: While $M \neq \emptyset$ set $M = M - \{i\}$ and $N = N - \{\sigma_i\}$ and return to step 1.

Step 4: Calculate x_i (in the any iteration method) from the equation σ_i.

Examples and Results

Example 1 *Let us take the matrix* $A \in R^{6\times 6}$*:*

$$\breve{A} = \begin{bmatrix} 101.002 & -4.537 & 18.14 & 200.543 & 856.541 & -633.473 \\ 830.315 & -345.747 & 334.625 & -861.904 & 6.386 & 514.616 \\ 698.748 & 770.724 & -107.767 & 144.902 & 256.671 & 844.855 \\ 368.74 & -905.051 & -593.629 & -323.017 & 132.668 & -521.816 \\ -491.357 & 408.077 & 470.494 & -700.024 & 776.917 & -573.267 \\ -537.963 & -433.094 & 894.934 & 492.262 & 852.806 & 642.922 \end{bmatrix},$$

So we have

$M_{1R} = 856541, M_{2R} = 861.904, M_{3R} = 844.855, M_{4R} = 905.051, M_{5R} = 776.917, M_{6R} = 894.934$

and

$M_{1C}, = 830.315, M_{2C} = 905.051, M_{3C} = 894.934, M_{4C} = 861.904, M_{5C} = 856.541, M_{6C} = 844.855$

hence $fd := 776.917$ and

$$\breve{A} = \begin{bmatrix} 101.002 & 4.537 & 18.14 & 200.543 & \underline{776.917} & 633.473 \\ \underline{776.917} & 345.747 & 334.625 & \underline{776.917} & 6.386 & 514.616 \\ 698.748 & 770.724 & 107.767 & 144.902 & 256.671 & \underline{776.917} \\ 368.74 & \underline{776.917} & 593.629 & 323.017 & 132.668 & 521.816 \\ 491.357 & 408.077 & 470.494 & 700.024 & \underline{776.917} & 573.267 \\ 537.963 & 433.094 & \underline{776.917} & 492.262 & \underline{776.917} & 642.922 \end{bmatrix},$$

fd do not exists in a permutation (see first and fifth columns). Again we have $[\breve{A}]_{pq} = [\breve{A}]_{32} = 770.724$, hence $fd := 770.724$ and

$$\breve{A} = \begin{bmatrix} 101.002 & 4.537 & 18.14 & 200.543 & \underline{770.724} & 633.473 \\ \underline{770.724} & 345.747 & 334.625 & \underline{770.724} & 6.386 & 514.616 \\ 698.748 & \underline{770.724} & 107.767 & 144.902 & 256.671 & \underline{770.724} \\ 368.74 & \underline{770.724} & 593.629 & 323.017 & 132.668 & 521.816 \\ 491.357 & 408.077 & 470.494 & 700.024 & \underline{770.724} & 573.267 \\ 537.963 & 433.094 & \underline{770.724} & 492.262 & \underline{770.724} & 642.922 \end{bmatrix},$$

again we have $[\breve{A}]_{pq} = [\breve{A}]_{54} = 700.024$, hence $fd := 700.024$ and

$$\breve{A} = \begin{bmatrix} 101.002 & 4.537 & 18.14 & 200.543 & \underline{700.024} & 633.473 \\ \underline{700.024} & 345.747 & 334.625 & \underline{700.024} & 6.386 & 514.616 \\ \underline{700.024} & \underline{700.024} & 107.767 & 144.902 & 256.671 & \underline{700.024} \\ 368.74 & \underline{700.024} & 593.629 & 323.017 & 132.668 & 521.816 \\ 491.357 & 408.077 & 470.494 & \underline{700.024} & \underline{700.024} & 573.267 \\ 537.963 & 433.094 & \underline{700.024} & 492.262 & \underline{700.024} & 642.922 \end{bmatrix},$$

so we have $fdet = 700.24 = \min\{|\alpha_{15}|, |\alpha_{21}|, |\alpha_{36}|, |\alpha_{42}|, |\alpha_{54}|, |\alpha_{63}|\}$.

Example 2: *Consider linear system:*

$$\begin{cases} 3x_1 - 2x_2 + 4x_3 + x_4 &= 14 \\ x_2 + 2x_3 - x_4 &= -3 \\ x_1 + 3x_2 + x_3 - 2x_4 &= -10 \\ 2x_1 + x_2 - 3x_3 - 5x_4 &= -18 \end{cases}$$

the exact solution is $(1,-2,1,3)^t$ and if we are used Jacobi and Gauss-Seidel methods by initial vector $(0,0,0,0)^t$, then the methods will fail. Jacobi method was given OVERFLOW in iteration 90 and Gauss-Seidel in iteration 42. If we use principal algorithm, we have $i = 2$ and $\sigma_2 = 3$, and then $i = 1, \sigma_1 = 1 : i = 3, \sigma_3 = 2 : i = 4, \sigma_4 = 4$, by as initial vector exact solution will receive in 15 number iteration for Gauss-Seidel method and in 52 number iteration for Jacobi method. Results shown in this table:

Initial Vec.	Jac.	Gau.	Jac. On PA	Gau. On PA
$(0,0,0,0)^t$	Fail(90)	Fail(42)	52	15
$(1,1,1,1)^t$	Fail(90)	Fail(43)	49	14
$(10,10,10,10)^t$	Fail(90)	Fail(41)	49	15

Also in example 1 iteration matrix(Jacobi and Gauss-Seidel) have eigenvalues:

$$\begin{pmatrix} 5.783697 \\ 0.352533 \pm 5.109815i \\ -3.208778 \pm 1.943071i \\ -0.071207 \end{pmatrix}$$

but in matrix PA which have $\sigma^1 = 5$, $\sigma^2 = 1$, $\sigma^3 = 6$, $\sigma^4 = 2$, $\sigma^5 = 4$, $\sigma^6 = 3$ have eigenvalues:

$$\begin{pmatrix} 0.904868 \pm 0.740244i \\ -0.246957 \pm 1.338781i \\ -.657911 \pm 0.209768i \end{pmatrix}$$

In the first case $\rho(A)$ and the second case $\rho(PA)$ = 1.346146.

CONCLUSION

In this paper, we present iteration algorithms to finding solution of $Ax = b$, which change $Ax = b$ to $PAx = Pb$ by commute rows of matrix A. The new algorithm (from fuzzy membership determinant) used to this method. The result show that by this permotation of rows of A we will have beter gonvergence of itration method.

ACKNOWLEDGEMENTS

Partial financial support from the Islamic Azad University, Khorasgan Branch, Are gratefully acknowledged.

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