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On the Domination Polynomials of Complete Partite Graphs

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Abstract: Let G be a simple graph of order n. The domination polynomial of G is the polynomial $D(G,x)=\sum_{i=1}^{n} d(G,i)x^i$, where d (G, i) is the number of dominating sets of G of size i. In this paper we study the domination polynomials of complete n-partite graphs.

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INTRODUCTION

Let G = (V, E) be a simple graph. The order of G is the number of vertices of G. A set $S\subseteq V$ is a dominating set if every vertex in $V\setminus S$ is adjacent to at least one vertex in S. The domination number γ (G) is the minimum cardinality of a dominating set in G. For a detailed treatment of this parameter, the reader is referred to [4]. Let D (G, i) be the family of dominating sets of G with cardinality i and let d (G, i) = |D(G, i)|.
The polynomial

$$D(G,x)=\sum_{i=1}^{|V(G)|} d(G,i)x^i$$

is defined as domination polynomial of G ([2]). A root of D (G, x) is called a domination root of G. For more information on this polynomial refer to [1-3].

For $t\in N$ and $n\geq 2$, a n-partite graph G whose V (G) can be partitioned into n non-empty subsets V_1, $V_2,\ldots,V_n$ such that each edge of G joins a vertex in V_i to a vertex in V_j for some distinct i, j in {1,2,…,n}. We call (V_1, $V_2,\ldots,V_n$) a n-partition of G. A complete n-partite graph is a n-partite graph with a n-partition (V_1, $V_2,\ldots,V_n$) such that every vertex in V_i is adjacent to every vertex in V_j for all distinct i, j in {1, 2,…,n}. Such a complete n-partite graph is denoted by K (m_1, $m_2,\ldots,m_n$) if $|V_i| = m_i$ for each i = 1, 2,…,n. A 2-partite graph is better known as a bipartite graph. The join of two graphs G_1 and G_2, denoted by G_1+G_2 is a graph with vertex set $V(G_1)\cup V(G_2)$ and edge set E $(G_1)\cup E(G_1)\cup\{uv|u\in V(G_1)$ and $v\in V(G_2)\}$. A graph is an empty graph if contain no edges. The empty graph of order n is denoted by O_n.

In this paper we study the domination polynomials and domination roots of n-partite graphs.

MAIN RESULTS

In this section we study the domination polynomials and the domination roots of n-partite graphs. First, we recall the following theorem which give a formula for the computation of the domination polynomial of join of two graphs.

Theorem 1: ([2]) Let G_1 and G_2 be graphs of orders n_1 and n_2, respectively. Then

$$D(G_1+G_2,x)=((1+x)^{n_1}-1)((1+x)^{n_2}-1) + D(G_1,x)+D(G_2,x)$$

Theorem 2

(i) $$D(K_{m,n},x)=((1+x)^m-1)((1+x)^n-1) + x^m + x^n$$

(ii) $$D(K_{m_1,\ldots,m_n},x)=\sum_{i=m_1}^{m_n} x^i + \sum_{i=2}^{n}((1+x)^{m_i}-1)((1+x)^{m_1+\ldots+m_{i-1}}-1)$$

Proof

(i) By applying Theorem 1 with $G_1 = O_n$ and $G_2 = O_m$, we have the result.
(ii) Since $K_{m_1,\ldots,m_n}=O_{m_1}+O_{m_2}+\ldots+O_{m_n}$ the result follows from part (i) and Induction hypothesis.

Theorem 3: For every natural number n, D ($K_{1,n}$, x) has exactly two real roots for odd n and exactly three real roots for even n.

Proof: Since

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$$D(K_{1,n}, x) = x^n + x(1+x)^n$$

it is suffices to prove that $x^{n-1}+(1+x)^n$ or $\frac{1}{x}+(1+\frac{1}{x})^n$ has exactly one real root for odd n and two real roots for even n. Put

$$f_n(x) = \frac{1}{x} + (1+\frac{1}{x})^n$$

Since the number of real roots of f_n (x) is equal to the number of real roots of

$$f_n(\frac{1}{x}) = (1+x)^n + x = 0$$

or $x^n+x-1 = 0$, we investigate the number of real roots of $g_n(x) = x^n+x-1 = 0$. Since $g_n(0) = -1<0$ and $g_n(1) = 1>0$, by intermediate value theorem, $g_n(x)$ has at least one real root in (0,1). Now, suppose that n is odd and $g_n(x)$ has two real roots. By Rolle's Theorem, there exists a real number c such that $g'_n(c)=nc^{n-1}+1=0$ and it is impossible because n-1 is an even. By a similar argument the theorem is proved for even n.

Corollary 1: For every even n, no nonzero real numbers is domination root of $K_{n,n}$.

Proof: By Theorem 2(i), we have

$$D(K_{n,n}, x) = ((1+x)^n - 1)^2 + 2x^n$$

If $D(K_{n,n}, x) = 0$, then $((1+x)^n - 1)^2 = -2x^n$. Obviously this equation does not have real nonzero solution for even n.

Using Maple, we observe that there are graphs such that all their domination roots except zero are complex ([1]). In Corollary 1, we observe that for every even n, no nonzero real numbers is domination root of $K_{n,n}$. So one of this kind graphs are $K_{n,n}$ for even n. Here, we state the following problem.

Problem: Characterize all graphs with no real domination root except zero.

REFERENCES

1. Akbari, S., S. Alikhani and Y.H. Peng, Characterization of graphs using Domination Polynomial. European Journal of Combinatorics, To Appear.
2. Alikhani, S. and Y.H. Peng, Introduction to Domination Polynomial of a Graph, Ars Combin., To Appear.
3. Alikhani, S. and Y.H. Peng, 2009. Dominating sets and Domination Polynomials of Paths. International Journal of Mathematics and Mathematical Science, Vol 2009, Article ID 542040.
4. Haynes, T.W., S.T. Hedetniemi and P.J. Slater, 1998. Fundamentals of Domination in Graphs, Marcel Dekker, NewYork.