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Numerical Solution of Logistic Differential Equations by Using the Laplace Decomposition Method

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Abstract: In this paper, the Laplace decomposition method is applied to some nonlinear differential equations to obtain their numerical solutions. The method is based on the Laplace transform and the nonlinear terms can be easily handled by the use of Adomian polynomials.

Key words: Laplace decomposition method · Logistic differential equation · Adomian polynomials

INTRODUCTION

During the last few years, so many mathematical methods that are aimed at solving non-linear ordinary differential equations (ODEs) and partial differential equations (PDEs) arising in engineering has appeared in the research literature [1-3]. However, most of them require a tedious analysis or a large computer memory to handle these problems. The Laplace transform is an elementary but useful technique for solving linear ordinary differential equations that is widely used by scientists and engineers for tackling linearized models. In fact, the Laplace transform is one of only a few methods that can be applied to linear systems with periodic or discontinuous driving inputs. Despite its great usefulness in solving linear problems, however, the Laplace transform is totally incapable of handling nonlinear equations because of the difficulties that are caused by the nonlinear terms. In order to extend its application, this paper using Adomian polynomials to decompose the nonlinear term, so that the solution can be obtained by iteration procedure. Various ways have been proposed recently to deal with these nonlinearities such as the Adomian decomposition method [4] and the Laplace decomposition algorithm that was proposed by Yusufoglu [5] in 2006. Furthermore, the Laplace transformation method is also combined with the well-known homotopy perturbation method [6] and the variational iteration method [7] to produce a highly effective technique for handling many nonlinear problems. These new developments, together with a host of other transform methods such as the differential transform [8-10] and the Sumudu transform [11], form the bulk of the collection of methods that are currently available for the solution of nonlinear differential equations. The main aim of this paper is to review some of the applications of the Laplace decomposition method to nonlinear problems in bioengineering and mechanical engineering. No attempt has been made, however, to solve nonlinear Logistic differential equations with this method. This paper considers the famous nonlinear Logistic population growth as a single species model as well as the Predator-Prey Models: Lotka-Volterra systems as an interacting species model to be governed by [12-14]. The main thrust of this technique is that the solution which is expressed as an infinite series converges fast to exact solutions.

Laplace Decomposition Method: This section discusses the use of the Laplace decomposition method for the solution of some nonlinear differential equations. To illustrate the basic concept of the technique, we consider a second-order nonhomogeneous nonlinear differential equation with given initial conditions of the general form

$$Du(t)+Ru(t)+Nu(t)=g(t) \tag{1}$$

$$u(0)=\lambda,\quad u'(0)=\eta, \tag{2}$$

Where:

λ and η are real constant, $D=\frac{\partial^2}{\partial t^2}$ is the second-order differential operator, R is the remaining linear operator, N

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represent a general non-linear differential operator and $g(t)$ is a source term. Using the Laplace decomposition method, first applying the Laplace transformation (L) on both sides

$$L[Du(t)] + L[Ru(t)] + L[Nu(t)] = L[g(t)], \tag{3}$$

by virtue of the differentiation property, yields

$$s^2 L[u(t)] - s\lambda - \eta + L[Ru(t)] + L[Nu(t)] = L[g(t)], \tag{4}$$

$$L[u(t)] = \frac{\lambda}{s} + \frac{\eta}{s^2} - \frac{1}{s^2} L[Ru(t)] + \frac{1}{s^2} L[g(t)] - \frac{1}{s^2} L[Nu(t)]. \tag{5}$$

The standard Laplace transformation method defines the solution $u(x,t)$ by the series

$$u = \sum_{n=0}^{\infty} u_n, \tag{6}$$

and decompose the nonlinear term into

$$Nu(t) = \sum_{n=0}^{\infty} A_n(u), \tag{7}$$

Where:
A_n are the Adomian polynomials of $u_0, u_1, u_2, u_3, \ldots, u_n$ that are given by

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[N\left(\sum_{n=0}^{\infty} \lambda^i u_i \right) \right]_{\lambda=0}, \quad n = 0, 1, 2, 3 \ldots\ldots \tag{8}$$

Substituting Eq. (6) and Eq. (7) into Eq. (5), we have

$$L\left[\sum_{n=0}^{\infty} u_n(t)\right] = \frac{\lambda}{s} + \frac{\eta}{s^2} - \frac{1}{s^2} L[Ru(t)] + \frac{1}{s^2} L[g(t)] - \frac{1}{s^2} L\left[\sum_{n=0}^{\infty} A_n(u)\right], \tag{9}$$

$$\sum_{n=0}^{\infty} L[u_n(t)] = \frac{\lambda}{s} + \frac{\eta}{s^2} - \frac{1}{s^2} L[Ru(t)] + \frac{1}{s^2} L[g(t)] - \frac{1}{s^2} L\left[\sum_{n=0}^{\infty} A_n(u)\right], \tag{10}$$

which gives, by comparing both sides of Eq. (10),

$$L[u_0(t)] = \frac{\lambda}{s} + \frac{\eta}{s^2} + \frac{1}{s^2} L[g(t)], \tag{11}$$

$$\begin{aligned} L[u_1(t)] &= -\frac{1}{s^2} L[Ru_0(t)] - \frac{1}{s^2} L[A_0(u)], \\ L[u_2(t)] &= -\frac{1}{s^2} L[Ru_1(t)] - \frac{1}{s^2} L[A_1(u)], \\ L[u_3(t)] &= -\frac{1}{s^2} L[Ru_2(t)] - \frac{1}{s^2} L[A_2(u)]. \end{aligned} \tag{12}$$

In general, the recursive relation is given by

$$L[u_{n+1}(t)] = -\frac{1}{s^2} L[Ru_n(t)] - \frac{1}{s^2} L[A_n(u)], \quad n \geq 0 \tag{13}$$

By taking the inverse Laplace transform from both sides of Eq. (11)-Eq. (13), one obtains

$$u_0(t) = H(t), \tag{14}$$

$$u_{n+1}(t) = -L^{-1}\left[\frac{1}{s^2} L[Ru_n(t)] + \frac{1}{s^2} L[A_n(u)]\right], \quad n \geq 0, \tag{15}$$

Where:
$H(t)$ is a function that arises from the source term and the prescribed initial conditions.

Logistic Differential Equation: Consider the logistic growth of a population to be a single species model governed by the equation [13].

$$\frac{dN}{dt} = rN(1 - N/K) \tag{16}$$

for some positive constants r and K. Here $N = N(t)$ represents the population of the species at time t, and $r(1 - N/K)$ is the per capita growth rate. K is the carrying capacity of the environment. We non-dimensionlize "Eq. (16)" by setting

$$u(r) = \frac{N(t)}{K}, \quad \tau = rt, \tag{17}$$

So that "Eq. (16)" becomes

$$\frac{du}{d\tau} = u(1 - u). \tag{18}$$

$$\text{If } N(0) = N_0 \text{ then } u(0) = N_0 / K. \tag{19}$$

Now consider the Lotka-Volterra system which is an interacting species Predator-Prey model governed by [12-14].

$$\begin{aligned} \frac{dN}{dt} &= N(a - bP) \\ \frac{dP}{dt} &= P(eN - d) \end{aligned} \tag{20}$$

For some constants a,b,c and d, where $N = N(t)$ and $P = P(t)$ are the prey and predator populations at time t respectively. We non-dimensionlize the system (20) by setting

$$u(\tau)=\frac{cN(t)}{d}, v(\tau)=\frac{bP(t)}{a}, \tau=at, \alpha=d/a \tag{21}$$

So that it becomes

$$\begin{aligned}\frac{du}{d\tau}&=u(1-v),\\ \frac{dv}{d\tau}&=\alpha v(1-u).\end{aligned} \tag{22}$$

Analysis of the Method for Single Species: The general form of "Eq. (18)" is

$$\frac{du}{d\tau}=u-f(u), \quad u(0)=\gamma, \tag{23}$$

Where:

$$f(u)=u^2 \tag{24}$$

Is a nonlinear function of u and we seek the solution u satisfying Eq. (23). By applying the aforesaid method subject to initial condition, we have

$$u(s)=\frac{N_0}{Ks}+\frac{1}{s}L\left[u-u^2\right], \tag{25}$$

Operating with Laplace inverse on both sides of Eq. (25) gives

$$u(\tau)=\frac{N_0}{K}+L^{-1}\left[\frac{1}{s}L\left[u-u^2\right]\right]. \tag{26}$$

Since a series form of the solution

$$u(\tau)=\sum_{n=0}^{\infty}u_n(\tau) \tag{27}$$

Is required by the Laplace decomposition method, Eq. (27) is substituted into Eq. (26) to produce

$$\sum_{n=0}^{\infty}u_n(\tau)=\frac{N_0}{K}+L^{-1}\left[\frac{1}{s}L\left[\sum_{n=0}^{\infty}u_n(\tau)-\sum_{n=0}^{\infty}A_n(u)\right]\right], \tag{28}$$

Where:

$A_n(u)$ is the Adomian polynomial [15] that represents a nonlinear term and we have

$$\sum_{n=0}^{\infty}A_n(u)=u^2(\tau). \tag{29}$$

The first few components of the Adomian polynomial are given below

$$\begin{aligned}&A_0(u)=u_0^{\,2}(\tau),\\ &A_1(u)=2u_0(\tau)u_1(\tau),\\ &A_2(u)=2u_0(\tau)u_2(\tau)+u_1^{\,2}(\tau),\\ &\vdots\end{aligned} \tag{30}$$

For numerical purposes, we have taken $N_0 = 2$ and $K = 1$. The recursive relation, which follows from Eq. (28), is given by

$$u_0(\tau)=\frac{N_0}{K}=2, \tag{31}$$

$$u_{n+1}(\tau)=L^{-1}\left[\frac{1}{s}L\left[u_n(\tau)-\sum_{n=0}^{\infty}A_n(u)\right]\right], \quad n\geq 0. \tag{32}$$

For $n = 0$, we have

$$\begin{aligned}u_1(\tau)&=L^{-1}\left[\frac{1}{s}L\left[u_0(\tau)-A_0(u)\right]\right],\\ u_1(\tau)&=L^{-1}\left[\frac{1}{s}L\left[\frac{N_0}{K}-\frac{N_0^{\,2}}{K^2}\right]\right],\\ u_1(\tau)&=L^{-1}\left[\frac{1}{s^2}\left(\frac{N_0}{K}-\frac{N_0^{\,2}}{K^2}\right)\right],\\ u_1(\tau)&=\left(\frac{N_0}{K}-\frac{N_0^{\,2}}{K^2}\right)\tau=-2\tau,\end{aligned} \tag{33}$$

and for n=1we have

$$\begin{aligned}u_2(\tau)&=L^{-1}\left[\frac{1}{s}L\left[u_1(\tau)-A_1(u)\right]\right],\\ u_2(\tau)&=\frac{N_0\tau^2}{2K}-\frac{3N_0^{\,2}\tau^2}{2K^2}+\frac{N_0^{\,3}\tau^2}{K^3}=3\tau^2.\end{aligned} \tag{34}$$

The solution in a series form is given by

$$\begin{aligned}u(\tau)&=u_0(\tau)+u_1(\tau)+u_2(\tau)+u_3(\tau)+u_4(\tau)+u_5(\tau)+\ldots\\ u(\tau)&=2-2\tau+3\tau^2-\frac{13\tau^3}{3}+\frac{25\tau^4}{4}-\frac{541\tau^5}{60}+\ldots.\end{aligned} \tag{35}$$

Example. Consider the general form of Eq. (22).

$$\begin{aligned}\frac{du}{d\tau}&=u-f(u,v),\\ \frac{dv}{d\tau}&=\alpha[g(u,v)-v],\end{aligned} \tag{36}$$

and the initial conditions

$$u(0)=\delta, v(0)=\beta, \tag{37}$$

for δ = 1.3 and β = 0.6. Setting α = 1 and

$$f(u,v)=g(u,v)=uv, \tag{38}$$

In a similar way we have

$$u(s)=\frac{1.3}{s}+\frac{1}{s}L\left[u(\tau)-u(\tau)v(\tau)\right],$$
$$v(s)=\frac{0.6}{s}+\frac{1}{s}L\left[\left[u(\tau)v(\tau)-v(\tau)\right]\right] \tag{39}$$

the inverse of Laplace transform implies that

$$u(\tau)=1.3+L^{-1}[\frac{1}{s}L\left[u(\tau)-u(\tau)v(\tau)\right]],$$
$$v(\tau)=0.6+L^{-1}[\frac{1}{s}L\left[u(\tau)v(\tau)-v(\tau)\right]]., \tag{40}$$

The recursive relation is given by

$$u_0(\tau)=1.3, \tag{41}$$

$$u_{n+1}(\tau)=L^{-1}\left[\frac{1}{s}L\left[u_n(\tau)-\sum_{n=0}^{\infty}A_n(u,v)\right]\right], \qquad n\geq 0, \tag{42}$$

and

$$v_0(\tau)=0.6, \tag{43}$$

$$v_{n+1}(\tau)=L^{-1}\left[\frac{1}{s}L\left[\sum_{n=0}^{\infty}A_n(u,v)-v_n(\tau)\right]\right], \qquad n\geq 0, \tag{44}$$

Where:

$A_n(u,v)$ are Adomian polynomials that represent nonlinear terms given by

$$\sum_{n=0}^{\infty}A_n(u,v)=u(\tau)v(\tau). \tag{45}$$

The first few components of the Adomian polynomials, for example, are

$$\begin{aligned}
&A_0(u,v)=u_0(\tau)v_0(\tau),\\
&A_1(u,v)=u_0(\tau)v_1(\tau)+u_1(\tau)v_0(\tau),\\
&A_2(u,v)=u_0(\tau)v_2(\tau)+u_1(\tau)v_1(\tau)+u_2(\tau)v_0(\tau),\\
&A_3(u,v)=u_0(\tau)v_3(\tau)+u_1(\tau)v_2(\tau)+u_2(\tau)v_1(\tau)+u_3(\tau)v_0(\tau)\\
&\vdots
\end{aligned} \tag{46}$$

and the recursive relations, from Eq. (42) and Eq. (44), are

Table 1: exhibits the comparison between LDM, ADM and VIM

τ	LDM	ADM	VIM
0	2	2	2
0.05	1.90699	1.90699	1.90698
0.1	1.8262	1.8262	1.82604
0.15	1.75535	1.75535	1.7547
0.2	1.69245	1.69245	1.6909
0.25	1.6354	1.6354	1.63293

Table 2: Exhibits the comparison between LDM, ADM and VIM solution of $u(\tau)$

τ	LDM	ADM	VIM
0	1.3	1.3	1.3
0.05	1.32598	1.32598	1.32594
0.1	1.35187	1.35187	1.35167
0.15	1.37757	1.37757	1.37704
0.2	1.40297	1.40297	1.40186
0.25	1.42797	1.42797	1.42598

Table 3: exhibits the comparison between LDM, ADM and VIM solution of $v(\tau)$

τ	LDM	ADM	VIM
0	0.6	0.6	0.6
0.05	0.609463	0.609463	0.609466
0.1	0.619878	0.619878	0.619899
0.15	0.631281	0.631281	0.631359
0.2	0.643711	0.643711	0.64391
0.25	0.657209	0.657209	0.657625

$$\begin{aligned}
&u_1(\tau)=L^{-1}\left[\frac{1}{s}L\left[u_0(\tau)-A_0(u,v)\right]\right],\\
&u_1(\tau)=L^{-1}\left[\frac{1}{s}L\left[u_0(\tau)-u_0(\tau)v_0(\tau)\right]\right],\\
&u_1(\tau)=0.52\tau,\\
&v_1(\tau)=L^{-1}\left[\frac{1}{s}L\left[A_0(u,v)-v_0(\tau)\right]\right],\\
&v_1(\tau)=L^{-1}\left[\frac{1}{s}L\left[u_0(\tau)v_0(\tau)-v_0(\tau)\right]\right],\\
&v_1(\tau)=0.18\tau,
\end{aligned} \tag{47}$$

$$\begin{aligned}
&u_2(\tau)=L^{-1}\left[\frac{1}{s}L\left[u_1(\tau)-A_1(u,v)\right]\right],\\
&u_2(\tau)=-0.13\tau^2,\\
&v_2(\tau)=L^{-1}\left[\frac{1}{s}L\left[A_1(u,v)-v_1(\tau)\right]\right],\\
&v_2(\tau)=0.1830\tau^2,\\
&u_3(\tau)=L^{-1}\left[\frac{1}{s}L\left[u_2(\tau)-A_2(u,v)\right]\right]\\
&u_3(\tau)=-0.1122\tau^3\\
&v_3(\tau)=L^{-1}\left[\frac{1}{s}L\left[A_2(u,v)-v_2(\tau)\right]\right]\\
&v_3(\tau)=0.0469\tau^3\\
&u_4(\tau)=L^{-1}\left[\frac{1}{s}L\left[u_3(\tau)-A_3(u,v)\right]\right]\\
&u_4(\tau)=-0.0497\tau^4\\
&v_4(\tau)=L^{-1}\left[\frac{1}{s}L\left[A_3(u,v)-v_3(\tau)\right]\right]\\
&v_4(\tau)=0.0099\tau^4\\
&\vdots
\end{aligned} \tag{48}$$

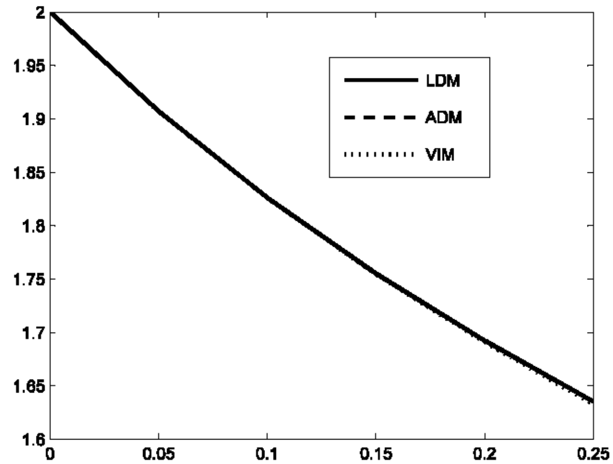

Fig. 1: Graphic comparison between LDM, ADM and VIM solution

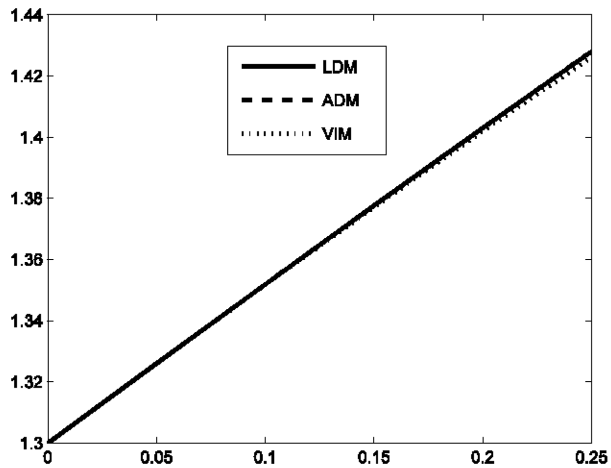

Fig. 2: Graphic Comparison between LDM, ADM and VIM solution of u

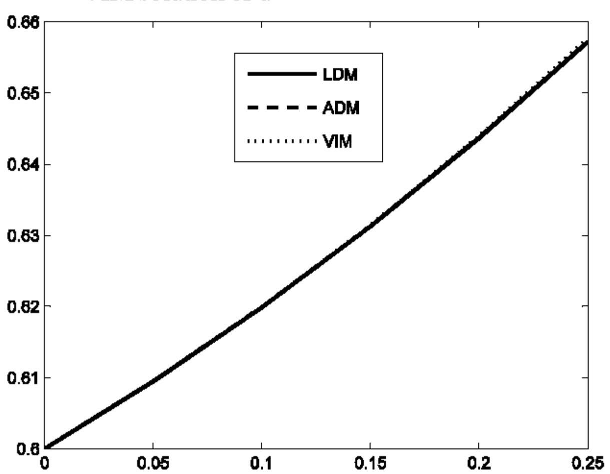

Fig. 3: Graphic comparison between LDM, ADM and VIM of v

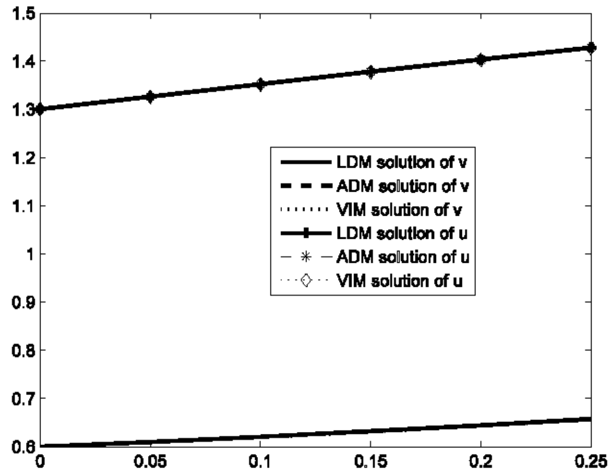

Fig. 4: Graphic Comparison between LDM, ADM and VIM of u and v

We have therefore obtained the following approximate solution to the initial problem:

$$u(\tau)=u_0(\tau)+u_1(\tau)+u_2(\tau)+u_3(\tau)+u_4(\tau)+.... \quad (49)$$

$$=1.3+0.52\tau-0.13\tau^2-0.1122\tau^3-0.0497\tau^4-....$$

$$v(\tau)=v_0(\tau)+v_1(\tau)+v_2(\tau)+v_3(\tau)+v_4(\tau)+.... \quad (50)$$

$$=0.6+0.18\tau+0.1830\tau^2+0.0469\tau^3+0.00997\tau^4+....$$

CONCLUSION

In this paper, the series solution of a logistic differential equation is obtained by using the Laplace decomposition m ethod (LDM). Comparisons are made between the Laplace decomposition method, the Adomian decomposition method and the variational iteration method. The present technique has closed agreement with ADM and VIM. This analysis does not exist in the present literature and it provides further support for the validity and soundness of the LDM.

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