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Smart Numerov Generalized Alternating Group Explicit (SNAGE-PR (2)) Scheme for 2-Point B.V. Problem

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Abstract: This paper considers the effect of smart Numerov generalized alternating group explicit (SNAGE-PR (2)) scheme in solution of two point boundary value problems also we have shown that the method is convergent. Here, it is shown that, by using this scheme the accuracy of solution is increased and most importantly the time of computation is decreased. Numerical results confirm the efficiency of our approach.

Key words: NUMAGE method . Generalized AGE method . SMAGE method . Tri-diagonal linear systems . 2-point B.V. problem

INTRODUCTION

Two point non-linear boundary value problems frequently occur in many complex mathematical modeling problems in science and engineering in which the solution of a linear or non-linear ordinary differential equation is required where the boundary conditions are given at two different points [1]. There are a variety of numerical strategies which can be applied i.e. [5, 7, 8]. Finite difference approximations have been used for solving the two-point boundary value problems:

$$-U'' + q(x)U = f(x) \qquad a \le x \le b \tag{1.1}$$

subject to boundary conditions,

$$U(a) = \alpha, \qquad U(b) = \beta \tag{1.2}$$

where α, β are real constants and q (x), f (x) real continuous functions with $q(x) \ge 0$.
We place a uniform mesh of size,

$$h = \frac{(b-a)}{(m+1)} \tag{1.3}$$

and the mesh points of the discrete problem are given by

$$x_i = a + ih, \qquad 0 \le i \le m+1 \tag{1.4}$$

Based on finite Taylor's series expansions and applying finite difference approximations to the equation (1.1) results as follows:

$$-u_{i-1} + 2g_i u_i - u_{i+1} = h^2 f_i, \qquad 1 \le i \le m \tag{1.5}$$

where $g_i = 1 + 0.5q_i h^2$ and the local truncation error is of order O (h^2). Equation (1.5) in matrix notation can be written as:

$$Au = b \tag{1.6}$$

where

$$u = [u_1, u_2, \ldots, u_{m-1}, u_m]^T \tag{1.7}$$

and

$$b = [\alpha + h^2 f_1, h^2 f_2, \ldots, h^2 f_{m-1}, \beta + h^2 f_m]^T$$

For solution of equations (1.6) SMAGE method is proposed although by using this method time consumption is reduced but truncation error of this method is of order O (h^2) [3]. Later Evans and Ahmad introduced NUMAGE method in which the truncation error of this method is of order O (h^4) but this method requires high time consumption [4].

In this paper, our scheme is based on the combination AGE-PR (2) and SMAGE methods and using of the Numerov's formula for the two-point boundary value problems subject to Dirichlet boundary conditions.

In section 2, we briefly describe generalized AGE method. In section 3, we introduce Numerov-AGE-PR

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(2) method also we have shown that the method is convergent and in section 4, we introduce our scheme, smart Numerov AGE-PR (2). Finally, several examples are presented which confirm the efficiency of the presented scheme.

GENERALIZED AGE METHOD

Evans [2] has introduced the alternating group explicit (AGE) method for the iterative solution of systems of linear equation, where $A_{m\times m}$ is tri-diagonal matrix:

$$A=\begin{pmatrix} 2g_1 & c_1 & & & \\ a_2 & 2g_2 & c_2 & & \\ & \ddots & \ddots & \ddots & \\ & & a_{m-1} & 2g_{m-1} & c_{m-1} \\ & & & a_m & 2g_m \end{pmatrix} \tag{2.1}$$

The AGE iterative method consists of splitting the matrix A into the form

$$A = G_1+G_2 \tag{2.2}$$

where

$$G_1=\begin{pmatrix} g_1 & c_1 & & & & & \\ a_2 & g_2 & & & & & \\ & & g_3 & c_3 & & & \\ & & a_4 & g_4 & & & \\ & & & & \ddots & & \\ & & & & & g_{m-1} & c_{m-1} \\ & & & & & a_m & g_m \end{pmatrix} \tag{2.3}$$

$$G_2=\begin{pmatrix} g_1 & & & & & & \\ & g_2 & c_2 & & & & \\ & a_3 & g_3 & & & & \\ & & & \ddots & & & \\ & & & & g_{m-2} & c_{m-2} & \\ & & & & a_{m-1} & g_{m-1} & \\ & & & & & & g_m \end{pmatrix} \tag{2.4}$$

and m is even

$$G_1=\begin{pmatrix} g_1 & & & & & \\ & g_2 & c_2 & & & \\ & a_3 & g_3 & & & \\ & & & \ddots & & \\ & & & & g_{m-1} & c_{m-1} \\ & & & & a_m & g_m \end{pmatrix} \tag{2.5}$$

$$G_2=\begin{pmatrix} g_1 & c_1 & & & & \\ a_2 & g_2 & & & & \\ & & \ddots & & & \\ & & & g_{m-2} & c_{m-2} & \\ & & & a_{m-1} & g_{m-1} & \\ & & & & & g_m \end{pmatrix} \tag{2.6}$$

where m is odd. For any iteration parameter $r \succ 0$, the AGE iterative scheme is as follows:

$$(rI+G_1)u^{(k+1/2)} = b+(rI-G_2)u^{(k)} \tag{2.7}$$

$$(rI+G_2)u^{(k+1)} = b+(rI-G_1)u^{(k+1/2)} \tag{2.8}$$

The iteration matrix is given by

$$T_r = (rI+G_2)^{-1}(rI-G_1)(rI+G_1)^{-1}(rI-G_2) \tag{2.9}$$

By using a similarity transformation, we have

$$\tilde{T}_r = (rI-G_1)(rI+G_1)^{-1}(rI-G_2)(rI+G_2)^{-1} \tag{2.10}$$

Since G_1 and G_2 matrices are positive definite, this implies their eigen-values are real and positive, i.e.

$$g_i \succ \frac{1}{2}(a_i+c_i), i=1,2,\ldots,m$$

It is shown that the AGE scheme is convergent [2]. The modification of equation (2-8) (AGE-PR (2) scheme) [3] is as follows:

$$(rI+G_1)u^{(k+1/2)} = b+(rI-G_2)u^{(k)} \tag{2.11}$$

$$(rI+G_2)u^{(k+1)} = 2ru^{(k+1/2)}-(rI-G_2)u^{(k)} \tag{2.12}$$

for any $r \succ 0$. By using equation (2.7) to express $G_1u^{(k+1/2)}$ in terms of $G_2u^{(k)}$, thereby saving on the evaluation of the right hand side vectors, the AGE-PR (2) scheme can be written in explicit form:

$$u^{(k+1/2)} = (rI+G_1)^{-1}[b+(rI-G_2)u^{(k)}] \tag{2.13}$$

$$u^{(k+1)} = (rI+G_2)^{-1}[2ru^{(k+1/2)}-(rI-G_2)u^{(k)}] \tag{2.14}$$

and the iteration matrix is given by:

$$T_r = (rI+G_2)^{-1}[2r(rI+G_1)^{-1}-I](rI-G_2) \tag{2.15}$$

The consistency and convergence of AGE-PR (2) scheme is proved [3]. Since G_1 and G_2 are non-symmetric, for the convergence, we need to show $\|T_r\|_2 \prec 1$. For this scheme,

$$\|T_r\|_2 = \|(rI+G_2)^{-1}[2r(rI+G_1)^{-1} - I](rI-G_2)\|_2 \qquad (2.16)$$

Let μ and ν be the respective eigen-values of G_1 and G_2. Since all the eigen-values of G_1 and G_2 are positive, then we have from (2.16)

$$\begin{aligned}\|T_r\|_2 &= \|(rI+G_2)^{-1}[2r(rI+G_1)^{-1} - I](rI-G_2)\|_2 \\ &= \left|\frac{1}{(r+\nu)}\left(\frac{2r}{(r+\mu)}-1\right)(r-\nu)\right| \qquad (2.17)\\ &= \max_{\mu,\nu}\frac{|(r-\mu)(r-\nu)|}{|(r+\mu)(r+\nu)|} \prec 1\end{aligned}$$

Thus, the AGE-PR (2) scheme is convergent.

NUMEROV-AGE-PR (2) METHOD

With the usual centered three point approximation, the truncation error is of order O (h^2). To increase accuracy in the numerical solution, the well known recurrence solution is called the Numerov formula [4] is applied which given by:

$$u_{i-1} - 2u_i + u_{i+1} = \frac{1}{12}h^2(u''_{i-1} + 10u''_i + u''_{i+1}) \qquad (3.1)$$

The finite difference Eq. (3.1) has been shown to have the truncation error of O (h^4). We write Eq. (1.1) as

$$U'' = q(x)U - f(x) \qquad (3.12)$$

Let $u''_i = U''(x_i)$, $q_i = q(x_i)$ and $f_i = f(x_i)$ then we have $u''_i = q_i u_i - f_i$, By substituting (3.2) in (3.1):

$$a_i u_{i-1} + 2g_i u_i + c_i u_{i+1} = w_i \qquad (3.3)$$

where

$$a_i = -1 + \frac{1}{12}h^2 q_{i-1}$$

$$g_i = 1 + \frac{5}{12}h^2 q_i$$

$$c_i = -1 + \frac{1}{12}h^2 q_{i+1}$$

and

$$w_i = \frac{1}{12}h^2(f_{i-1} + 10f_i + f_{i+1}). \quad 1 \le i \le m$$

The Eq. (3.3) can be written in the matrix form

$$Au = b$$

where

$$u = [u_1, u_2, \ldots, u_{m-1}, u_m]^T$$

and

$$b = [w_1 - a_1, w_2, \ldots, w_{m-1}, w_m - c_m]^T$$

The matrix A is given by:

$$A = \begin{pmatrix} 2g_1 & c_1 & & & \\ a_2 & 2g_2 & c_2 & & \\ & \ddots & \ddots & \ddots & \\ & & a_{m-1} & 2g_{m-1} & c_{m-1} \\ & & & a_m & 2g_m \end{pmatrix} \qquad (3.4)$$

where G_1 and G_2 are introduced in section 2. It can be shown that for the non-symmetric matrices, the NAGE-PR (2) as same as AGE method [4] is convergent provided that all the eigen-values of the matrices are positive.

The eigen-values of G_1 and G_2 are $\lambda_1 = g_i$ for i = 1,m and those which are given by the determinant equation

$$\det\begin{pmatrix} \lambda - g_i & c_i \\ a_{i+1} & \lambda - g_{i+1}\end{pmatrix} = 0$$

which has the roots

$$\lambda_2 = \frac{1}{2}(g_i + g_{i+1}) - \frac{1}{2}\sqrt{(g_i - g_{i+1})^2 + 4a_{i+1}c_i}$$

and

$$\lambda_3 = \frac{1}{2}(g_i + g_{i+1}) + \frac{1}{2}\sqrt{(g_i - g_{i+1})^2 + 4a_{i+1}c_i}$$

Both λ_1 and λ_3 are positive and we will show that λ_2 is also positive. This can be shown as follows:

$$\frac{1}{2}(g_i + g_{i+1}) = 1 + \frac{5}{24}h^2(q_i + q_{i+1})$$

$$(g_i - g_{i+1})^2 = \frac{25}{144}h^4(q_i - q_{i+1})^2$$

$$4a_{i+1}c_i = 4 - \frac{1}{3}h^2(q_i + q_{i+1}) + \frac{1}{36}h^4 q_i q_{i+1}$$

By neglecting all terms which contain h^4, then we have

$$\frac{1}{2}\sqrt{(g_i - g_{i+1})^2 + 4a_{i+1}c_i} = \frac{1}{2}\sqrt{4 - \frac{1}{3}h^2(q_i + q_{i+1})} \le 1$$

Hence, λ_2 is positive. Therefore NAGE-PR (2) method is convergent. Now, by using the NAGE-PR (2) method, we can determine $u^{(k+1)}$ in two steps from the explicit form of (2.11) and (2.12). Assume that m is even. In programming the AGE method, we need to store the arrays g_i, α_i, β_i, c_i and a_i, but not d_i as this value may be assigned as a variable. We now write the algorithm for the Numerov-AGE-PR (2) formula:

Algorithm 3.1: The NAGE-PR (2) method for the model problem (1.1)

1. Set $u_i^{(k)} = 0, i = 0,\ldots,m+1, a_1 = 0, c_m = 0$
2. To compute $u^{(k+1/2)}$. Set i = 1

While i≤m-1, compute

$$\begin{aligned} r_1 &= b_i - a_i u_{i-1}^{(k)} + \beta_i u_i^{(k)} \\ r_2 &= b_{i+1} + \beta_{i+1} u_{i+1}^{(k)} - c_{i+1} u_{i+2}^{(k)} \\ d &= 1/(\alpha_i \alpha_{i+1} - a_{i+1} c_i) \\ u_i^{(k+1/2)} &= (\alpha_{i+1} r_1 - c_i r_2) d \\ u_{i+1}^{(k+1/2)} &= (-a_{i+1} r_1 + \alpha_i r_2) d \\ i &= i+2. \end{aligned}$$

3. To compute $u^{(k+1)}$. Set i = 2

$$u_1^{(k+1)} = (2ru_1^{(k+1/2)} + \beta_1 u_1^{(k)}) / \alpha_1$$

while i≤m-2, compute

$$\begin{aligned} r_1 &= 2ru_i^{(k+1/2)} - \beta_i u_i^{(k)} + c_i u_{i+1}^{(k)} \\ r_2 &= 2ru_{i+1}^{(k+1/2)} + a_{i+1} u_i^{(k)} - \beta_{i+1} u_{i+1}^{(k)} \\ d &= 1/(\alpha_i \alpha_{i+1} - a_{i+1} c_i) \\ u_i^{(k+1)} &= (\alpha_{i+1} r_1 - c_i r_2) d \\ u_{i+1}^{(k+1)} &= (-a_{i+1} r_1 + \alpha_i r_2) d \\ i &= i+2. \\ u_m^{(k+1)} &= (2ru_m^{(k+1/2)} - \beta_m u_m^{(k)}) / \alpha_m \end{aligned}$$

4. Repeat step 2 and step 3 until convergence is achieved.

SMART NUMEROV AGE-PR (2) (SNAGE-PR (2)) SCHEME

Evans and Ahmad [3] introduced a same form of the AGE method, which is called the smart AGE (SMAGE) method. In this section, we will consider an efficient form of the SMAGE method which is called the smart Numerov generalized alternating group explicit (SNAGE-PR (2)) scheme.

Consider two point boundary value problem (1.1) subject to the boundary condition, by applying Numerov formula to equation (1.1), we have

$$a_i u_{i-1} + 2 g u_i + c_i u_{i+1} = w_i \tag{4.1}$$

Where

$$\begin{aligned} a_i &= -1 + \frac{1}{12} h^2 q_{i-1},\ g_i = 1 + \frac{5}{12} h^2 q_i,\ c_i = -1 + \frac{1}{12} h^2 q_{i+1}, \\ w_i &= \frac{1}{12} h^2 (f_{i-1} + 10 f_i + f_{i+1}), 1 \le i \le m \end{aligned} \tag{4.2}$$

The equation (4.1) can be written in the matrix form Au = b, where matrix A is introduced in (3.4). Now by using AGE-PR (2) method, we can determine $u^{(k+1)}$ in two steps from the explicit form of (2.13) and (2.14). Assume m be even for any $r \succ 0$, then the block sub-matrices ($rI+G_1$), ($rI+G_2$), ($rI-G_1$) and ($rI-G_2$) have the form

$$\tilde{G} = \begin{pmatrix} \alpha_i & c_i \\ a_{i+1} & \alpha_{i+1} \end{pmatrix} \text{and} \tilde{G} = \begin{pmatrix} \beta_i & -c_i \\ -a_{i+1} & \beta_{i+1} \end{pmatrix} \tag{4.3}$$

where, α_i = r+g and β_i = r-g_i.. The inverse of $\tilde{G}$ is given by

$$\tilde{G}^{-1} = d_i \begin{pmatrix} \alpha_{i+1} & -c_i \\ -a_{i+1} & \alpha_i \end{pmatrix} \tag{4.4}$$

where

$$d_i = 1/(\alpha_i \alpha_{i+1} - a_{i+1} c_i), \quad i = 1,\ldots,m/2 \tag{4.5}$$

We will predict that this new scheme save time as the idea involved is eliminating evaluating two similar terms on the right hand sides of the AGE-PR(2) scheme and also, the truncation error is of order $O(h^4)$. The two similar terms in (2-11) and (2-12) equations is $(rI-G_2)u^{(k)}$ and we let this term to be ϕ. The evaluation and saving of ϕ depends on whether the problem is linear or non-linear. We expect SNAGE-PR(2) scheme for linear problems to save two multiplications and one addition for every iteration, while for non-linear problems it is expected to save one multiplication and one addition.

The SNAGE-PR (2) algorithm where the values of g_i, c_i, a_i and w_i is introduced in section 3 are as follows:

1. Set $u_i^{(k)} = 0, i = 0,\ldots,m+1, a_1 = 0, c_m = 0$
2. To compute g_i, α_i and β_i
   For i = 1 to m
   Compute g_i, α_i = r+g_i and β_i = r-g_i
3. To compute $\phi = (rI-G_2)u^{(k)}$ set i = 1
   while i≤m-1, compute

$$\begin{aligned} \phi_i &= -c_i u_{i-1}^{(k)} + \beta_i u_i^{(k)} \\ \phi_{i+1} &= \beta_{i+1} u_{i+1}^{(k)} - a_{i+1} u_{i+2}^{(k)} \\ i &= i+2 \end{aligned}$$

4. To compute $u^{(k+1/2)}$. Set i=1
   while i≤m-1, compute

$$
\begin{aligned}
r_1 &= b_i + \phi_i \\
r_2 &= b_{i+1} + \phi_{i+1} \\
d &= 1/(\alpha_i \alpha_{i+1} - a_{i+1} c_i) \\
u_i^{(k+1/2)} &= (\alpha_{i+1} r_1 - c_i r_2) d \\
u_{i+1}^{(k+1/2)} &= (-a_{i+1} r_1 + \alpha_i r_2) d \\
i &= i + 2
\end{aligned}
$$

5. For i = 1,2,…,m compute $\phi_i = -\phi_i + 2ru_i^{(k+1/2)}$
6. To compute $u^{(k+1)}$

$$u_1^{(k+1)} = \phi_1 / \alpha_1$$

While i≤m-2

$$
\begin{aligned}
d &= 1/(\alpha_i \alpha_{i+1} - a_{i+1} c_i) \\
u_i^{(k+1)} &= (\alpha_{i+1} \phi_i - c_i \phi_{i+1}) d \\
u_{i+1}^{(k+1)} &= (-a_{i+1} \phi_i + \alpha_i \phi_{i+1}) d \\
i &= i + 2. \\
u_m^{(k+1)} &= \phi_m / \alpha_m .
\end{aligned}
$$

7. Repeat step 2 and step 6 until convergence is achieved.

It can be noticed that for linear problems, the values of g_i, α_i and β_i are unchanged for each iteration, so these values can be computed outside of the iteration loop, while for non-linear problems, the values of g_i, α_i and β_i varies at every iteration. Therefore, the computations for these values must be kept within the iteration loop. Consequently, the intermediate computation of ϕ_i must also be carried out within the loop.

NUMERICAL EXPERIMENTS

In this section, we experimentally investigate the SNAGE-PR (2) scheme on 3 problems. The obtained results are presented as follows. Each problem will be concerned with the speed (CPU time) and the number of iterations. Also, for showing the accuracy of each problem, we show in following tables in spatial case, truncation error of approximate solutions and exact solutions. The time is measured initially from the initialization of $u^{(0)}$ until the solution converges to $u^{(k)}$, where k is the number of iterations. The numerical results are carried out to a tolerance 10^{-7}. In Table 1, 3 and 5, we show the truncation error of SMAGE and SNAGE-PR (2) schemes for m = 20. Also, in Table 2, 4 and 6, we compare the CPU time for different schemes. The results show the agreement between the computational work, with the CPU time and the order of accuracy for each scheme.

Table 1: Truncation error of test 1 for m = 20

x_i	Exact solution	SMAGE	SNAGE-PR(2)
0.05	0.0334514	2.392132620e-06	5.364835381e-10
0.14	0.1010901	7.056703549e-06	6.159778243e-10
0.24	0.17094315	1.136600085e-05	2.417612333e-09
0.33	0.2445091	1.507634138e-05	4.343818055e-09
0.43	0.32332021	1.793298953e-05	5.116507495e-09
0.52	0.40895636	1.966500044e-05	2.906721475e-09
0.62	0.50305937	1.997921351e-05	3.848415031e-09
0.71	0.60734791	1.855304343e-05	1.473337063e-08
0.81	0.72363312	1.502563331e-05	2.438709568e-08
0.9	0.85383503	8.973926487e-06	4.968759575e-09

Table 2: The CPU time taken of test 1 for m=20

N	iter	AGE-PR (o2)	NAGE-PR (2)	SNAGE-PR (2)
40	117	0.016	0.016	0.000
80	227	0.188	0.219	0.016
160	539	0.922	1.047	0.157
320	1225	4.547	4.828	1.032
640	2710	20.393	20.893	8.845
1280	5971	106.936	105.030	70.522

Table 3: Truncation error of test 2 for m=20

x_i	Exact solution	SMAGE	SNAGE-PR(2)
2.05	0.00946441	1.522625180e-06	8.693270118e-09
2.14	0.02513426	3.801252304e-06	3.707214737e-08
2.24	0.03666715	5.226231900e-06	6.401131418e-08
2.33	0.04427736	5.970508739e-06	8.886972230e-08
2.43	0.04814557	6.164594261e-06	1.098716873e-07
2.52	0.0484252	5.911050214e-06	1.277883651e-07
2.62	0.04524732	5.248581249e-06	1.023458561e-07
2.71	0.03872463	4.281171425e-06	7.568308488e-08
2.81	0.02895455	3.058736949e-06	4.900061859e-08
2.9	0.01602171	1.621570964e-06	2.394452103e-08

Test 1: Consider

$$
\begin{aligned}
-U'' + U &= -x, \quad 0 \le x \le 1 \\
U(0) &= 0, \qquad U(1) = 1
\end{aligned}
\tag{5.1}
$$

The exact solution to this problem is given by

$$U(x) = \frac{2e}{e^2 - 1}(e^x - e^{-x}) - x \tag{5.2}$$

Test 2: Consider

Table 4: The CPU time taken of test 2

N	iter	AGE-PR (2)	SMAGE-PR (2)	NAGE-PR (2)	SNAGE-PR (2)
40	86	0.015	0.000	0.016	0.000
80	203	0.251	0.016	0.322	0.031
160	476	1.000	0.172	1.115	0.172
320	1063	4.654	0.985	5.989	1.000
640	2318	23.298	7.845	25.443	7.844
1280	5030	98.776	59.880	114.255	63.240

Table 5: Truncation error of test 3 for m=20

x_i	Exact solution	SMAGE	SNAGE-PR (2)
0.05	1.00113314	1.35428620e-05	6.703152255e-09
0.14	1.01015208	3.219603907e-05	1.548261519e-10
0.24	1.02794422	4.701081702e-05	3.947242089e-09
0.33	1.05402185	5.655029950e-05	5.030522976e-09
0.43	1.08766258	6.084489038e-05	2.700470625e-08
0.52	1.12792037	6.025606521e-05	3.521910785e-08
0.62	1.17363989	5.511044565e-05	2.837088631e-08
0.71	1.22347413	4.591552883e-05	3.684526129e-08
0.81	1.27590511	3.318848197e-05	5.938221692e-09
0.9	1.32926725	1.764767009e-05	3.792957060e-10

Table 6: The CPU time taken of test 3

N	iter	AGE-PR (2)	SMAGE-PR (2)	NAGE-PR (2)	SNAGE-PR (2)
40	127	0.032	0.016	0.046	0.015
80	303	0.266	0.031	0.266	0.032
160	710	1.359	0.219	1.155	0.203
320	1606	6.407	1.328	5.727	1.344
640	3553	33.129	12.298	30.347	12.752
1280	7845	191.205	104.183	190.653	104.090

$$U'' = \frac{2}{x^2}U - \frac{1}{x}, \qquad 2 \le x \le 3$$
$$U(2) = 0, \qquad U(3) = 0 \tag{5.3}$$

The exact solution to this problem is given by

$$U(x) = \frac{1}{38}(19x - 5x^2 - \frac{36}{x}) \tag{5.4}$$

Test 3: Consider

$$U'' = -U + 2\cos x, \qquad 0 \le x \le 1 \tag{5.5}$$

The exact solution to this problem is given by

$$U(x) = \cos x + x\sin x \tag{5.6}$$

CONCLUSIONS

In this paper, we applied the idea of SMAGE method [3] and combination of SMAGE method with AGE-PR (2) scheme. Our main approach focused on convergence of scheme and also we have shown that our scheme SNAGE-PR (2) is fast and more accurate. The SNAGE-PR (2) method is shown to be numerically stable when using the Numerov formula in order to attain a greater accuracy. Also, the significant point our scheme is the capable on working with two parameters. The SNAGE-PR (2) is simple to implement and efficient in terms of CPU time. Finally, the numerical results show that the greater accuracy is achieved when using the Numerov formula.

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