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Redefined Fuzzy K-algebras

M. Akram[1], *K. H. Dar* [2], *B. L. Meng*[3] *and Y. L. Liu*[4]

[1] PUCIT, University of the Punjab, Old Campus, P. O. Box 54000, Lahore, Pakistan
[2] Department of Mathematics, G. C. University Lahore, P.O. Box 54000, Pakistan
[3] Department of Basic Courses, Xian University of Science and Technology, P. R. China
[4] Department of Mathematics, Wuyi University, Wuyishan, Fujian 354300, P. R. China

Abstract: We first introduce a new sort of fuzzy K-subalgebra called, an $(\prec, \prec \vee \Upsilon)^*$- fuzzy K-subalgebra of K-algebras using the notion of an anti fuzzy point and its besideness to and non-quasi-coincidence with a fuzzy set, and investigate some of its interesting properties. Then we introduce the concept of an anti fuzzy K-subalgebra with thresholds. Finally, we discuss generalization of an $(\prec, \prec \vee \Upsilon)^*$- fuzzy K-subalgebra.

Key words: K-algebra, Besidesness, non-quasi-coincidence, $(\prec, \prec \vee \Upsilon)^*$- fuzzy subalgebra, $(\prec, \prec \vee \Upsilon_m)^*$- fuzzy K-subalgebras

INTRODUCTION

The notion of a K-algebra $(G, \cdot, \odot, e)$ was first introduced by Dar and Akram [1] in 2003 and published in 2005. A K-algebra is an algebra built on a group $(G, \cdot, e)$ by adjoining an induced binary operation $\odot$ on G which is attached to an abstract K-algebra $(G, \cdot, \odot, e)$. This system is, in general non-commutative and non-associative with a right identity e, if $(G, \cdot, e)$ is non-commutative. For a given group G, the K-algebra is proper if G is not an elementary abelian 2-group. Thus, a K-algebra is abelian and non-abelian purely depends on the base group G. Dar and Akram further renamed *a K-algebra on a group G* as a $K(G)$-algebra [2] due to its structural basis G. The $K(G)$-algebras have been characterized by their left and right mappings in [2] when group is abelian.

In 1965, Zadeh [3] introduced the notion of a fuzzy subset of a set as a method for representing uncertainty. Since then it has become a vigorous area of research in different domains such as engineering, medical science, social science, physics, statistics, graph theory, artificial intelligence, signal processing, multiagent systems, pattern recognition, robotics, computer networks, expert systems, decision making, automata theory. On the other hand, Murali [4] proposed a definition of a fuzzy point belonging to a fuzzy subset under a natural equivalence on a fuzzy set. The idea of quasi-coincidence of a fuzzy point with a fuzzy set, which is mentioned in [5] played a vital role to generate some different types of fuzzy subgroups. A new type of fuzzy subgroups, $(\in, \in \vee q)$-fuzzy subgroups, was introduced in earlier paper Bhakat and Das [6] by using the combined notions of *belongingness* and *quasi-coincidence* of fuzzy point and fuzzy set. In fact, $(\in, \in \vee q)$-fuzzy subgroup is an important and useful generalization of Rosenfeld's fuzzy subgroup. Following this idea, Akram introduced the notion of $(\in, \in \vee q)$-fuzzy ideals of a K-algebra in [7]. In this paper, we introduce a new sort of fuzzy K-subalgebra called, an $(\prec, \prec \vee \Upsilon)^*$- fuzzy K-subalgebra of K-algebras using the notion of an anti fuzzy point and its besideness to and non-quasi-coincidence with a fuzzy set, and investigate some of its interesting properties. We then introduce the concept of an anti fuzzy K-subalgebra with thresholds. In particular, we discuss generalization of an $(\prec, \prec \vee \Upsilon)^*$- fuzzy K-subalgebra. Some recent results obtained by Akram [8] and Saeid-Jun [9] are extended. The definitions and terminologies that we used in this paper are standard. For other notations, terminologies and applications, the readers are referred to [10-16].

PRELIMINARIES

In this section we review some elementary facts that are necessary for this paper.

Definition 1 [1] Let $(G, \cdot, e)$ be a group in which each non-identity element is not of order 2. Then a K- *algebra* is a structure $\mathcal{K} = (G, \cdot, \odot, e)$ on a group G in which induced binary operation $\odot : G \times G \to G$ is defined by $\odot(x, y) = x \odot y = x \,.\, y^{-1}$ and satisfies the following axioms:

(K1) $(x \odot y) \odot (x \odot z) = (x \odot ((e \odot z) \odot (e \odot y))) \odot x$,

(K2) $x \odot (x \odot y) = (x \odot (e \odot y)) \odot x$,

(K3) $(x \odot x) = e$,

(K4) $(x \odot e) = x$,

(K5) $(e \odot x) = x^{-1}$

for all $x, y, z \in G$.

Corresponding Author: Muhammad Akram, m.akram@pucit.edu.pk

Definition 2 [10] A K-algebra $\mathcal{K}$ is called abelian if and only if $x \odot (e \odot y) = y \odot (e \odot x)$ for all $x, y \in G$.

If a K-algebra $\mathcal{K}$ is abelian, then the axioms $(K1)$ and $(K2)$ can be written as:

$(\overline{K1})$ $(x \odot y) \odot (x \odot z) = z \odot y$.

$(\overline{K2})$ $x \odot (x \odot y) = y$.

A nonempty subset H of a K-algebra $\mathcal{K}$ is called a *subalgebra* [1] of the K-algebra $\mathcal{K}$ if $a \odot b \in H$ for all $a, b \in H$. Note that every subalgebra of a K-algebra $\mathcal{K}$ contains the identity e of the group G. A mapping $f : \mathcal{K}_1 = (G_1, \cdot, \odot, e_1) \to \mathcal{K}_2 = (G_2, \cdot, \odot, e_2)$ of K-algebras is called a *homomorphism* [10] if $f(x \odot y) = f(x) \odot f(y)$ for all $x, y \in \mathcal{K}_1$. We note that if f is a homomorphism, then $f(e) = e$.
We now introduce the following notions:

Definition 3 A *fuzzy set* $\gamma : G \to [0, 1]$ is called an *anti fuzzy subalgebra* of $\mathcal{K}$ if

$$\gamma(e) \leq \gamma(x), \quad \gamma(x \odot y) \leq \max\{\gamma(x), \gamma(y)\}$$

hold for all $x, y \in G$.

We give the following Lemma without proof.

Lemma 4 Let γ be a fuzzy set in $\mathcal{K}$. Then γ is an anti fuzzy subalgebra of $\mathcal{K}$ if and only if

$$L(\gamma; t) = \{x \in G : \gamma(x) \leq t\}$$

is a subalgebra of $\mathcal{K}$ for all $t \in (0, 1]$.

Definition 5 For a family $\{\mu_i : i \in \Lambda\}$ of fuzzy sets of $\mathcal{K}$, we define for all $x \in \mathcal{K}$:

(1) $\bigcup_{i\in\Lambda} \mu_i(x) = (\sup_{i\in\Lambda} \mu_i)(x) = \sup_{i\in\Lambda} \mu_i(x)$,

(2) $\bigcap_{i\in\Lambda} \mu_i(x) = (\inf_{i\in\Lambda} \mu_i)(x) = \inf_{i\in\Lambda} \mu_i(x)$.

Definition 6 [8, 9] A fuzzy set γ in a set G of the form

$$\gamma(y) = \begin{cases} t \in [0, 1), & \text{if } y = x \\ 1, & y \neq x \end{cases}$$

is called an *anti fuzzy point* with support x and value t and is denoted by x_t. An anti fuzzy point x_t is said to "*besides to*" a fuzzy set γ, written as $x_t \prec \gamma$ if $\gamma(x) \leq t$. An anti fuzzy point x_t is said to be "*non-quasicoincident with*" a fuzzy set γ, denoted by $x_t \Upsilon \gamma$ if $\gamma(x) + t \leq 1$.

The idea of non-quasi-coincidence of an anti fuzzy point with a fuzzy set play a vital role to generate some different types of fuzzy subalgebras, called $(\alpha, \beta)^*$-fuzzy subalgebras, for $\alpha, \beta \in \{\prec, \Upsilon, \prec \vee \Upsilon, \prec \wedge \Upsilon\}$.
Notations: The following notations will be used in this paper:

- $x_t \prec \gamma$ and $x_t \Upsilon \gamma$ will be denoted by $x_t \prec \wedge \Upsilon \gamma$.
- $x_t \prec \gamma$ or $x_t \Upsilon \gamma$ will be denoted by $x_t \prec \vee \Upsilon \gamma$.
- The symbol $\overline{\prec \wedge \Upsilon}$ means neither $\prec$ nor Υ hold.

$(\prec, \prec \vee \Upsilon)^*-$ FUZZY K-SUBALGEBRA

Definition 7 A fuzzy set γ in G is called an $(\prec, \prec \vee \Upsilon)^*-$fuzzy K-subalgebra of $\mathcal{K}$ if the following conditions are satisfied:

(3) $x_s \prec \gamma \Rightarrow e_s \prec \vee \Upsilon \gamma$,

(4) $x_s, y_t \prec \gamma \Rightarrow (x \odot y)_{\max(s,t)} \prec \vee \Upsilon \gamma$

for all $x, y \in G$, $s, t \in [0, 1)$.

Example 8 Consider the K-algebra $\mathcal{K}$=$(S_3, \cdot, \odot, e)$ on the symmetric group S_3=$\{e, a, b, x, y, z\}$ where $e = (1)$, $a = (123)$, $b = (132)$, $x = (12)$, $y = (13)$, $z = (23)$, and $\odot$ is given by the following Cayley's table:

$\odot$	e	x	y	z	a	b
e	e	x	y	z	b	a
x	x	e	a	b	z	y
y	y	b	e	a	x	z
z	z	a	b	e	y	x
a	a	z	x	y	e	b
b	b	y	z	x	a	e

Define a fuzzy set γ in $\mathcal{K}$ by

$$\gamma(t) = \begin{cases} 0.1, & \text{if } t = e, \\ 0.3, & \text{if } t = a, \\ 0.4, & \text{if } t = b, x \\ 0.6, & \text{if } t = y, z. \end{cases}$$

By routine computations, it is easy to see that γ is an $(\prec, \prec \vee \Upsilon)^*$-fuzzy subalgebra of $\mathcal{K}$.

Theorem 9 Let γ be a fuzzy set in a algebra $\mathcal{K}$. Then γ is an $(\prec, \prec \vee \Upsilon)^*-$fuzzy subalgebra of $\mathcal{K}$ if and only if it satisfies the following

$$\gamma(x \odot y) \leq \max(\gamma(x), \gamma(y), 0.5), \quad for\ all\ x, y \in G.$$

Proof: Let x, $y \in G$. We consider the following two cases:

(i) $\max(\gamma(x), \gamma(y)) > 0.5$.

(ii) $\max(\gamma(x), \gamma(y)) \leq 0.5$.

Case (i): Assume that $\gamma(x \odot y) > \max(\gamma(x), \gamma(y), 0.5)$. Then $\gamma(x \odot y) > \max(\gamma(x), \gamma(y))$. Take s such that $\gamma(x \odot y) > s > \max(\gamma(x), \gamma(y))$. Then $x_s \prec \gamma, y_s \prec \gamma$, but $(x \odot y)_s \overline{\prec \vee \Upsilon} \gamma$, a contradiction.
Case (ii): Assume that $\gamma(x \odot y) > 0.5$. Then $x_{0.5}, y_{0.5} \prec \gamma$ but $(x \odot y)_{0.5} \overline{\prec \vee \Upsilon} \gamma$, a contradiction.

Conversely let x_s, $y_t \prec \gamma$, then $\gamma(x) \leq s$, $\gamma(y) \leq t$. Now, we have

$$\gamma(x \odot y) \leq \max(\gamma(x), \gamma(y), 0.5) \leq \max(s, t, 0.5).$$

If $\max(s,t) < 0.5$, then $\gamma(x \odot y) \leq 0.5 \Rightarrow \gamma(x \odot y) + \max(s,t) < 1$. On the other hand, if $\max(s,t) \geq 0.5$, then $\gamma(x \odot y) \leq \max(s,t)$. Hence $(x \odot y)_{\max(s,t)} \prec \vee\Upsilon\gamma$. □

Theorem 10 Let γ be an $(\prec, \prec \vee\Upsilon)^*$-fuzzy subalgebra of $\mathcal{K}$.

(i) If there exists $x \in G$ such that $\gamma(x) \leq 0.5$, then $\gamma(e) \leq 0.5$.

(ii) If $\gamma(e) > 0.5$, then γ is an $(\prec, \prec)^*$-fuzzy subalgebra of $\mathcal{K}$.

Proof:

(i) Let $x \in G$ such that $\gamma(x) \leq 0.5$.

$$\gamma(x^{-1}) = \max(\gamma(x), 0.5) = 0.5.$$

Hence

$$\begin{aligned} \gamma(e) &= \gamma(x \odot x)) = \gamma(x.x^{-1}) \\ &\leq \max(\gamma(x), \gamma(x^{-1}), 0.5) = 0.5. \end{aligned}$$

(ii) $\gamma(e) > 0.5 \Rightarrow \gamma(x) > 0.5$ for all $x \in G$. Thus we conclude that:

$$\gamma(x \odot y) \leq \max(\gamma(x), \gamma(y)), \forall\, x,\ y\ \in G.$$

Hence γ is an $(\prec, \prec)^*$-fuzzy subalgebra of $\mathcal{K}$.

□

Theorem 11 Let γ be a fuzzy set of $\mathcal{K}$. Then γ is an $(\prec, \prec\ \vee\Upsilon)^*$-fuzzy subalgebra of $\mathcal{K}$ if and only if the nonempty set $L(\gamma; t)$, $t \in [0,1)$ is a subalgebra of $\mathcal{K}$.

Proof: Assume that γ is an $(\prec, \prec\ \vee\Upsilon)^*$ fuzzy subalgebra of $\mathcal{K}$ and let $t \in [0,1)$. If $x, y \in L(\gamma; t)$, then $\gamma(x) \leq t$ and $\gamma(y) \leq t$. Thus, it follows that

$$\gamma(x \odot y) \leq \max(\gamma(x), \gamma(y), 0.5) \leq \max(t, 0.5) = t,$$

and so $x \odot y \in L(\gamma; t)$. This shows that $L(\gamma; t)$ is a subalgebra of $\mathcal{K}$.
Conversely, let γ be a fuzzy set such that $L(\gamma; t) = \{x \in G : \gamma(x) \leq t\}$ is a subalgebra of $\mathcal{K}$, for all $t \in [0,1)$. If there exist $x, y \in G$ such that $\gamma(x \odot y) > \max(\gamma(x), \gamma(y), 0.5)$, then we can take $t \in (0.5, 1)$ such that

$$\gamma(x \odot y) > t > \max(\gamma(x), \gamma(y), 0.5).$$

Thus $x, y \in L(\gamma; t)$ and $t > 0.5$, and but $x \odot y \notin L(\gamma; t)$, which contradicts to the assumption that all $L(\gamma; t)$ are subalgebras. Therefore,

$$\gamma(x \odot y) \leq \max(\gamma(x), \gamma(y), 0.5).$$

Hence γ is an $(\prec, \prec \vee\Upsilon)^*$ fuzzy subalgebra of $\mathcal{K}$. □

Theorem 12 Let γ be a fuzzy set in a K-algebra $\mathcal{K}$. Then $L(\gamma; t)$, $t \in [0.5, 1)$ is a subalgebra of $\mathcal{K}$ if and only if

$$\min(\gamma(x \odot y), 0.5) \leq \max(\gamma(x), \gamma(y)), \forall\, x,\ y\ \in G.\ (A)$$

Proof: Suppose that $L(\gamma; t)$ is a subalgebra of $\mathcal{K}$. Let x, $y \in G$, $\min(\gamma(x \odot y), 0.5) > \max(\gamma(x), \gamma(y)) = t$, then $t \in [0.5, 1)$, $\gamma(x \odot y) > t$, $x \prec L(\gamma; t)$ and $y \prec L(\gamma; t)$. Since $x, y \prec L(\gamma; t)$ and $L(\gamma; t)$ is a subalgebra of $\mathcal{K}$, so $x \odot y \prec L(\gamma; t)$ or $\gamma(x \odot y) \leq t$, a contradiction with $\gamma(x \odot y) > t$.
Conversely, suppose that condition (A) holds. Assume that $t \in [0.5, 1)$, $x, y \prec L(\gamma; t)$. Then

$$\begin{aligned} 0.5 &> t \geq \max(\gamma(x), \gamma(y)) \\ &\geq \min(\gamma(x \odot y), 0.5) \\ &\Rightarrow \gamma(x \odot y) \leq t, \end{aligned}$$

and so $x \odot y \prec L(\gamma; t)$. This shows that $L(\gamma; t)$ is a subalgebra of $\mathcal{K}$.
□

Theorem 13 Let $G_0 \supset G_1 \supset \cdots \supset G_n = G$ be a strictly decreasing chain of a subalgebras of a K-algebra $\mathcal{K}$, then there exists $(\prec, \prec \vee\Upsilon)^*$-fuzzy subalgebra γ of $\mathcal{K}$ whose level subalgebras are precisely the members of the chain with $\gamma_1 = G_0$.

Proof: Let $\{t_i : t_i \in [0.5, 1), i = 1, 2, \cdots, n\}$ be such that $t_1 > t_2 > t_3 > \cdots > t_n$. Let $\gamma : G \to [0,1]$ defined by

$$\gamma(x) = \begin{cases} t, & \text{if } x = 0, \\ n, & \text{if } x = 0, x \in G_0 \\ t_1, & \text{if } x \in G_1 \setminus G_0, \\ t_2, & \text{if } x \in G_2 \setminus G_1, \\ \vdots & \\ t_n, & \text{if } x \in G_n \setminus G_{n-1}. \end{cases}$$

Let $x, y \in G$. If $x \odot y \in G_0$, then from Theorem 3.3, we have

$$\gamma(x \odot y) \leq 0.5 \leq \max(\gamma(x), \gamma(y), 0.5).$$

On the other hand, If $x \odot y \notin G_0$, then there exists i, $1 \leq i \leq n$ such that $x \odot y \in G_i \setminus G_{i-1}$ so that $\gamma(x \odot y) = t_i$. Now there exists $j(\geq i)$ such that $x \in G_j$ or $y \in G_j$. If x, $y \in G_k (k < i)$, then G_k is a subalgebra of $\mathcal{K}$, $x \odot y \in G_k$ which contradicts $x \odot y \notin G_{i-1}$. Thus

$$\gamma(x \odot y) \leq t_i \geq t_j \leq \max(\gamma(x), \gamma(y), 0.5).$$

Hence γ is an $(\prec, \prec\ \vee\Upsilon)^*$-fuzzy subalgebra of $\mathcal{K}$. It follows from the contradiction of γ that $\gamma_{1.0} = G_0, \gamma_{t_i} = G_i$ for $i = 1, 2, \cdots, n$. This completes the proof. □

Definition 14 An $(\prec, \prec \vee\Upsilon)^*$-fuzzy subalgebra of $\mathcal{K}$ is called *proper* if Im γ has at least two elements. Two $(\prec, \prec \vee\Upsilon)^*$- fuzzy subalgebras γ_1 and γ_2 are said to be *equivalent* if they have the same family of level subalgebras.

Theorem 15 A proper $(\prec, \prec \vee \Upsilon)^*$-fuzzy subalgebra of $\mathcal{K}$ such that cardinality of $\{\gamma(x) : \gamma(x) > 0.5\} \leq 2$ can be expressed as the union of two proper non-equivalent $(\prec, \prec \vee \Upsilon)^*$-fuzzy subalgebras of $\mathcal{K}$.

Proof: Let γ be a proper $(\prec, \prec \vee \Upsilon)^*$-fuzzy subalgebra of $\mathcal{K}$ with $\{\gamma(x) : \gamma(x) > 0.5\} = t_1, t_2, \cdots, t_n\}$ where $t_1 < t_2 < \cdots < t_n$ and $n \geq 2$. Then

$$\gamma_{0.5} \subseteq \gamma_{t_1} \subseteq \cdots \subseteq \gamma_{t_n} = G$$

is the chain of $(\prec, \prec \vee \Upsilon)^*$- subalgebras of γ. Define two fuzzy sets $\gamma_1, \gamma_2 \geq \gamma$ defined by

$$\mu_1(x) = \begin{cases} t_1, & \text{if } x \in \gamma_{t_1}, \\ t_2, & \text{if } x \in \gamma_{t_2} \setminus \gamma_{t_1}, \\ \vdots & \\ t_n, & \text{if } x \in \gamma_{t_n} \setminus \gamma_{t_{n-1}}, \end{cases}$$

$$\mu_2(x) = \begin{cases} \gamma(x), & \text{if } x \in \gamma_{0.5}, \\ n, & \text{if } x \in \gamma_{t_2} \setminus \gamma_{0.5}, \\ t_3, & \text{if } x \in \gamma_{t_3} \setminus \gamma_{t_2}, \\ \vdots & \\ t_n, & \text{if } x \in \gamma_{t_n} \setminus \gamma_{t_{n-1}}, \end{cases}$$

respectively, where $t_3 > n > t_2$. Then γ_1 and γ_2 are $(\prec, \prec \vee \Upsilon)^*$-fuzzy subalgebras of $\mathcal{K}$ with

$$\gamma_{t_1} \subseteq \gamma_{t_2} \subseteq \cdots \subseteq \gamma_{t_n}$$

and

$$\gamma_{t_{0.5}} \subseteq \gamma_{t_2} \subseteq \cdots \subseteq \gamma_{t_n}$$

being respectively chains of $(\Upsilon, \prec \vee \Upsilon)^*$-fuzzy subalgebras. Hence γ can be expressed as the union of two proper non-equivalent $(\prec, \prec \vee \Upsilon)^*$-fuzzy subalgebras of $\mathcal{K}$. □

Theorem 16 Let $\{\gamma_i : i \in \Lambda\}$ be a family of $(\prec, \prec)^*$-fuzzy subalgebra of $\mathcal{K}$. Then $\gamma = \bigcup_{i\in\Lambda} \gamma_i$ is an $(\prec, \prec)^*$-fuzzy subalgebra of $\mathcal{K}$.

Proof: Let $x_s \prec \gamma$ and $y_t \prec \gamma$, where $s, t \in [0, 1)$. Then $\gamma(x) \leq s$ and $\gamma(y) \leq t$. Thus we have $\gamma_i(x) \leq s$ and $\gamma_i(y) \leq t$ for all $i \in \Lambda$. Hence $\gamma_i(x \odot y) \leq \max(s, t)$. Therefore, $\gamma(x \odot y) \leq \max(s, t)$, which implies that $(x \odot y)_{\max\{s,t\}} \prec \gamma$. This completes the proof. □

Theorem 17 Let $\{\gamma_i : i \in \Lambda\}$ be a family of $(\prec, \prec \vee \Upsilon)^*$-fuzzy subalgebras of $\mathcal{K}$. Then $\gamma := \bigcap_{i\in\Lambda} \gamma_i$ is an $(\prec, \prec \vee \Upsilon)^*$-fuzzy subalgebra of $\mathcal{K}$.

Proof: By Theorem 9, we have $\gamma(x \odot y) \leq \max(\gamma(x), \gamma(y), 0.5)$, and hence

$$\begin{aligned} \gamma(x \odot y) &= \inf_{i\in\Lambda} \gamma_i(x \odot y) \\ &\leq \inf_{i\in\Lambda} \max\{\gamma_i(x), \gamma_i(y), 0.5\} \\ &= \max\{\inf_{i\in\Lambda} \gamma_i(x), \inf_{i\in\Lambda} \gamma_i(y), 0.5\} \\ &= \max\{\bigcap_{i\in\Lambda} \gamma_i(x), \bigcap_{i\in\Lambda} \gamma_i(y), 0.5\} \\ &= \max\{\gamma(x), \gamma(y), 0.5\}. \end{aligned}$$

The verification is analogous for other condition and we omit the details. By Theorem 9, it follows that γ is an $(\prec, \prec \vee \Upsilon)^*$-fuzzy subalgebra of $\mathcal{K}$. □

Remark 1. If $\{\gamma_i : i \in \Lambda\}$ is a family of $(\prec, \prec \vee \Upsilon)^*$-fuzzy subalgebras of $\mathcal{K}$. Is $\gamma = \bigcup_{i\in\Lambda} \gamma_i$ is an $(\prec, \prec \vee \Upsilon)^*$-fuzzy subalgebra of $\mathcal{K}$? When? The following example shows that it is not an $(\prec, \prec \vee \Upsilon)^*$-fuzzy subalgebra in general.

Example 18 Consider the K-algebra $\mathcal{K}$=$G, \cdot, \odot, e)$, where $G = \{e, a, a^2, a^3\}$ is the cyclic group of order 4 and $\odot$ is given by the following Cayley's table:

$\odot$	e	a	a^2	a^3
e	e	a^3	a^2	a
a	a	e	a^3	a^2
a^2	a^2	a	e	a^3
a^3	a^3	a^2	a	e

If we define fuzzy sets $\gamma_1, \gamma_2 : G \to [0, 1]$ by putting

$$\gamma_1(x) := \begin{cases} 0.6 & \text{if } x = e, \\ 1 & \text{if } x = a^3, \\ 0 & \text{if } x \in \{a, a^2\}. \end{cases}$$

$$\gamma_2(x) := \begin{cases} 0.3 & \text{if } x = e, \\ 1 & \text{if } x = a^3, \\ 0 & \text{if } x \in \{a, a^2\}. \end{cases}$$

then both γ_1 and γ_2 are $(\prec, \prec \vee \Upsilon)^*$-fuzzy subalgebras of $\mathcal{K}$, but $\gamma_1 \cup \gamma_2$ is not an $(\prec, \prec \vee \Upsilon)^*$-fuzzy subalgebra of $\mathcal{K}$ since

$$\begin{aligned} 1 &= \max(\gamma_1(a^3), \gamma_2(a^3)) \\ &= (\gamma_1 \cup \gamma_2)(a^3) \\ &= (\gamma_1 \cup \gamma_2)(a^1 \odot a^2) \\ &\leq \min((\gamma_1 \cup \gamma_2)(a), (\gamma_1 \cup \gamma_2)(a^2), 0.5) \\ &= \min(0, 0, 0.5) = 0. \end{aligned}$$

Theorem 19 Let $\{\gamma_i : i \in \Lambda\}$ be a family of $(\prec, \prec \vee \Upsilon)^*$-fuzzy subalgebras of $\mathcal{K}$ such that $\gamma_i \subseteq \gamma_j$ or $\gamma_j \subseteq \gamma_i$ for all $i, j \in \Lambda$. Then $\gamma := \bigcup_{i\in\Lambda} \gamma_i$ is an $(\prec, \prec \vee \Upsilon)^*$-fuzzy subalgebra of L.

Proof: By Theorem 9, we have $\gamma(x \odot y) \leq \max(\gamma(x), \gamma(y), 0.5)$, and hence

$$\begin{aligned} \gamma(x \odot y) &= \sup_{i\in\Lambda} \gamma_i(x \odot y) \\ &\leq \sup_{i\in\Lambda} \max\{\gamma_i(x), \gamma_i(y), 0.5\} \\ &= \max\{\sup_{i\in\Lambda} \gamma_i(x), \sup_{i\in\Lambda} \gamma_i(y), 0.5\} \\ &= \max\{\bigcup_{i\in\Lambda} \gamma_i(x), \bigcup_{i\in\Lambda} \gamma_i(y), 0.5\} \\ &= \max\{\gamma(x), \gamma(y), 0.5\}. \end{aligned}$$

It is not difficult to see that

$$\sup_{i\in\Lambda}\max\{\gamma_i(x),\gamma_i(y),0.5\} \geq \bigcup_{i\in\Lambda}\max(\gamma_i(x),\gamma_i(y),0.5).$$

Suppose that

$$\sup_{i\in\Lambda}\max\{\gamma_i(x),\gamma_i(y),0.5\} \neq \bigcup_{i\in\Lambda}\max(\gamma_i(x),\gamma_i(y),0.5),$$

then there exists s such that

$$\begin{aligned}&\sup_{i\in\Lambda}\max\{\gamma_i(x),\gamma_i(y),0.5\} > s\\ &\quad > \bigcup_{i\in\Lambda}\max(\gamma_i(x),\gamma_i(y),0.5).\end{aligned}$$

Since $\gamma_i \subseteq \gamma_j$ or $\gamma_j \subseteq \gamma_i$ for all $i, j \in \Lambda$, there exists $k \in \Lambda$ such that $s > \max(\gamma_k(x), \gamma_k(y), 0.5)$. On the other hand, $\max(\gamma_i(x), \gamma_i(y), 0.5) > s$ for all $i \in \Lambda$, a contradiction. Hence

$$\begin{aligned}&\sup_{i\in\Lambda}\max\{\gamma_i(x),\gamma_i(y),0.5\}\\ &= \max(\textstyle\bigcup_{i\in\Lambda}\gamma_i(x),\bigcup_{i\in\Lambda}\gamma_i(y),0.5)\\ &= \max\{\gamma(x),\gamma(y),0.5\}\end{aligned}$$

The verification is analogous for other condition and we omit the details. By Theorem 9, it follows that γ is an $(\prec, \prec \vee \Upsilon)^*$-fuzzy subalgebra of $\mathcal{K}$. □

Remark 20 From above discussion we know that: $L(\gamma;\, t)$ is a subalgebra of $\mathcal{K}$ when (i) $t \in [0,\, 1)$ (ii) $t \in [0,\, 0.5)$ (iii) $t \in [0.5,\, 1)$. An obvious question is whether μ is a kind of anti fuzzy subalgebra or not when $L(\gamma;\, t)(\neq \emptyset)$(e.g. $m_1,\, m_2 \in [0, 1]$ and $(m_1 < m_2)$. Based on this discussion, We can extend the concept of an anti fuzzy K-subalgebra with thresholds in the following way:

Definition 21 Let $m_1, m_2 \in [0, 1]$ and $m_1 < m_2$. If γ is a fuzzy set of a algebra $\mathcal{K}$, then γ is called an *anti fuzzy subalgebra with thresholds* (m_1, m_2) if

(5) $\min(\gamma(e), m_1) \leq \max(\gamma(x), m_2)$

(6) $\min(\gamma(x \odot y), m_1) \leq \max(\gamma(x), \gamma(y), m_2)$

for all $x, y \in G$.

Theorem 22 A fuzzy set γ of algebra $\mathcal{K}$ is an anti fuzzy subalgebra with thresholds (m_1, m_2) of $\mathcal{K}$ if and only if $L(\gamma; t)(\neq \emptyset)$, for $t \in (m_1, m_2]$, is a subalgebra of $\mathcal{K}$.

Proof: Assume that γ is an anti fuzzy subalgebra with thresholds (m_1, m_2) of $\mathcal{K}$. Let $x,\, y \in L(\gamma; t)$. Then $\gamma(x) \leq t$ and $\gamma(y) \leq t$, $t \in (m_1, m_2]$. Then it follows that

$$\begin{aligned}\min(\gamma(x\odot y), m_1) &\leq \max(\gamma(x),\gamma(y),m_2)\\ &= t \Longrightarrow \gamma(x\odot y)\leq t,\end{aligned}$$

and hence $x \odot y \in L(\gamma; t)$. This shows that $L(\gamma; t)$ is a subalgebra of $\mathcal{K}$.

Conversely, assume that γ is a fuzzy set such that $L(\gamma; t) \neq \emptyset$ is a subalgebra of $\mathcal{K}$ for $m_1, m_2 \in [0, 1]$ and $m_1 < m_2$. Suppose that $\min(\gamma(x \odot y), m_1) > \max(\gamma(x), \gamma(y), m_2) = t$, then $\gamma(x \odot y) > t$, $x \in L(\gamma; t)$, $y \in L(\gamma; t)$, $t \in (m_1, m_2]$. Since x, $y \in L(\gamma; t)$ and $L(\gamma; t)$ are subalgebras, $x \odot y \in L(\gamma; t)$, i.e., $\gamma(x \odot y) \leq t$. This is a contradiction. The verification is analogous for other condition and we omit the details. This completes the proof. □

Theorem 23 Let $f : \mathcal{K}_1 \to \mathcal{K}_2$ be an epimorphism of K-algebras. If γ is an anti fuzzy subalgebra with thresholds (m_1, m_2) in $\mathcal{K}_2$, then $f^{-1}(\gamma)$ is an anti fuzzy subalgebra with thresholds (m_1, m_2) in $\mathcal{K}_1$, where

$$f^{-1}(\gamma)(x) = \gamma(f(x)) \quad for\ all\ \ x \in \mathcal{K}_1.$$

Proof: Let x, $y \in \mathcal{K}_1$, then

$$\begin{aligned}&\min(f^{-1}(\gamma)(x\odot y), m_1) = \min(\gamma(f(x\odot y)), m_1)\\ &= \min(\gamma(f(x)\odot f(y)), m_1)\\ &\leq \max(\gamma(f(x)), \gamma(f(y)), m_2)\\ &= \max(f^{-1}(\gamma)(x), f^{-1}(\gamma)(y), m_2).\end{aligned}$$

The verification is analogous for other condition and we omit the details. Hence $f^{-1}(\gamma)$ is an anti fuzzy subalgebra with thresholds (m_1, m_2) in $\mathcal{K}_1$. □

Theorem 24 Let $f : \mathcal{K}_1 \to \mathcal{K}_2$ be an onto homomorphism of K-algebras. If γ is an anti fuzzy subalgebra with thresholds (m_1, m_2) in $\mathcal{K}_1$, then $f(\gamma)$ is an anti fuzzy subalgebra with thresholds (m_1, m_2) in $\mathcal{K}_2$, where

$$f(\gamma)(y) := \inf\{\gamma(x) \mid f(x) = y \quad for\ all\ y \in \mathcal{K}_2\}.$$

Proof: Let y_1, $y_2 \in \mathcal{K}_2$. Then

$$\begin{aligned}&\min(f(\gamma)(y_1\odot y_2), m_1) = \min(\inf\{\gamma(x_1\odot x_2), m_1)\\ &\mid f(x_1\odot x_2) = y_1\odot y_2\}, m_1)\\ &= \inf\{\min(\gamma(x_1\odot x_2), m_1) \mid f(x_1\odot x_2) = y_1\odot y_2\}\\ &\leq \inf\{\max(\gamma(x_1), \gamma(x_2), m_1) \mid f(x_1) = y_1, f(x_2) = y_2\}\\ &= \max(\inf\{\gamma(x_1) \mid f(x_1) = y_1\},\\ &\inf\{\gamma(x_2) \mid f(x_2) = y_2\}, m_2)\\ &= \max(f(\gamma)(y_1), f(\gamma)(y_2), m_2).\end{aligned}$$

The verification is analogous for other condition and we omit the details. Hence $f(\gamma)$ is an anti fuzzy subalgebra with thresholds (m_1, m_2) in $\mathcal{K}_2$. □

$(\prec, \prec \vee \Upsilon_m)^*-$ FUZZY K-SUBALGEBRA

Let m be an element of (0, 1]unless otherwise specified. By $x_t\ \Upsilon_m\ \gamma$, we mean $\gamma(x) + t + m < 1$. The notation $x_t \prec \vee\ \Upsilon_m\ \gamma$ means $x_t \prec \gamma$ or $x_t\ \Upsilon_m\ \gamma$.

Definition 25 A fuzzy set γ in G is called an $(\prec, \prec \vee \Upsilon_m)^*-$fuzzy K-subalgebra of $\mathcal{K}$ if the following conditions are satisfied:

(7) $x_s \prec \gamma \Rightarrow e_s \prec \vee\Upsilon_m\gamma$,

(8) $x_s, y_t \prec \gamma \Rightarrow (x \odot y)_{\max(s,t)} \prec \vee\Upsilon_m\gamma$

for all $x, y \in G$, $s, t \in [0, 1)$.

Note that an $(\prec, \prec \vee\Upsilon_m)^*-$fuzzy K-subalgebra with $m = 0$ is called $(\prec, \prec \vee\Upsilon)^*-$fuzzy K-subalgebra.

Example 26 Consider the K-algebra $\mathcal{K} = (G, \cdot, \odot, e)$, where $G = \{e, a, a^2, a^3, a^4\}$ is the cyclic group of order 5 and $\odot$ is given by the following Cayley's table:

$\odot$	e	a	a^2	a^3	a^4
e	e	a^4	a^3	a^2	a
a	a	e	a^4	a^3	a^2
a^2	a^2	a	e	a^4	a^3
a^3	a^3	a^2	a	e	a^4
a^4	a^4	a^3	a^2	a	e

Define a fuzzy set γ in G defined by

$$\gamma(x) = \begin{cases} 0.07, & \text{if x =e,} \\ 0.18, & \text{otherwise.} \end{cases}$$

By routine computations, we can easily check that γ is an $(\prec, \prec \vee\Upsilon_{0.4})^*$-fuzzy subalgebra of $\mathcal{K}$.

The following proposition is trivial.

Proposition 27 Every $(\prec, \prec)^*$-fuzzy subalgebra is an $(\prec, \prec \vee\Upsilon_m)^*$-fuzzy subalgebra.

Theorem 28 A fuzzy set γ in $\mathcal{K}$ is an $(\prec, \prec \vee\Upsilon_m)^*$-fuzzy subalgebra of $\mathcal{K}$ if and only if it satisfies the following assertion for all $x, y \in G$:

$$\gamma(x \odot y) \leq \max\{\gamma(x), \gamma(y), \frac{1-m}{2}\}. \qquad (B)$$

Proof: Let γ be an $(\prec, \prec \vee q_m)^*$-fuzzy subalgebra of $\mathcal{K}$. Assume that (B) is not valid. Then there exist $a, b \in G$ such that

$$\gamma(a \odot b) > \max\{\gamma(a), \gamma(b)), \frac{1-m}{2}\}.$$

If $\max(\gamma(a), \gamma(b) > \frac{1-m}{2}$, then $\gamma(a \odot b) > \max(\gamma(a), \gamma(b))$. Thus

$$\gamma(a \odot b) > t \geq \max\{\gamma(a), \gamma(b)\} \quad \text{for some } t \in [0, 1).$$

It follows that $a_t \prec \gamma$ and $b_t \prec \gamma$, but $(a \odot b)_t \overline{\prec} \gamma$, a contradiction. Moreover, $\gamma(a \odot b) + t > 2t > 1 - m$, and so $(a \odot b)_t \overline{q_m} \gamma$. Consequently $(a \odot b)_t \overline{\prec \vee\Upsilon_m}\, \gamma$, a contradiction. On the other hand, If $\max(\gamma(a), \gamma(b)) \leq \frac{1-m}{2}$, then $\gamma(a) \leq \frac{1-m}{2}$, $\gamma(b) \leq \frac{1-m}{2}$ and $\gamma(a \odot b) > \frac{1-m}{2}$. Thus $a_{\frac{1-m}{2}} \prec \gamma$, $b_{\frac{1-m}{2}} \prec \gamma$, but $(a \odot b)_{\frac{1-m}{2}} \overline{\prec} \gamma$. Also

$$\gamma(a \odot b) + \frac{1-m}{2} > \frac{1-m}{2} + \frac{1-m}{2} = 1 - m,$$

i.e., $(a \odot b)_{\frac{1-m}{2}} \overline{\Upsilon_m} \gamma$. Hence $(a \odot b)_{\frac{1-m}{2}} \overline{\prec \vee\Upsilon_m}\, \gamma$, a contradiction. So (B) is valid.

Conversely, assume that γ satisfies (B). Let $x, y \in G$ and $t_1, t_2 \in [0, 1)$ be such that $x_{t_1} \prec \gamma$ and $y_{t_2} \prec \gamma$. Then

$$\gamma(x \odot y) \leq \max(\gamma(x), \gamma(y), \frac{1-m}{2}) \geq \max(t_1, t_2, \frac{1-m}{2}).$$

Assume that $t_1 \geq \frac{1-m}{2}$ or $t_2 \geq \frac{1-m}{2}$. Then $\gamma(x) \leq \max(t_1, t_2))$, which implies that $(x \odot y)_{\max(t_1,t_2)} \prec \gamma$. Now suppose that $t_1 < \frac{1-m}{2}$ and $t_2 < \frac{1-m}{2}$. Then $\gamma(x \odot y) \leq \frac{1-m}{2}$, and thus

$$\gamma(x \odot y) + \max(t_1, t_2) > \frac{1-m}{2} + \frac{1-m}{2} = 1 - m,$$

i.e., $(x \odot y)_{\max(t_1,t_2)}\ \Upsilon_m \gamma$. Hence $(x \odot y)_{\max(t_1,t_2)}\ \Upsilon_m \gamma$, and consequently, γ is an $(\prec, \prec \vee\Upsilon_m)^*$-fuzzy subalgebra of $\mathcal{K}$. □

Taking $m = 0$ in Theorem 28, we obtain the following corollary.

Corollary 29 A fuzzy set γ in $\mathcal{K}$ is an $(\prec, \prec \vee\Upsilon)^*$-fuzzy subalgebra of $\mathcal{K}$ if and only if it satisfies the following assertion for all $x, y \in G$:

$$\gamma(x \odot y) \leq \max\{\gamma(x), \gamma(y), 0.5\}.$$

Theorem 30 Let γ be a fuzzy set of a K-algebra $\mathcal{K}$. Then the non-empty level set $L(\gamma; t)$ is a subalgebra of $\mathcal{K}$ for all $t \in [\frac{1-m}{2}, 1)$ if and only if γ satisfies the following assertion for all $x, y \in G$:

$$\min(\gamma(x \odot y), \frac{1-m}{2}) \leq \max(\gamma(x), \gamma(y).$$

Proof: Let $t \in [\frac{1-m}{2}, 1)$ such that $L(\gamma; t)(\neq \emptyset)$ is a subalgebra of $\mathcal{K}$. Assume that $\min(\gamma(x \odot y), \frac{1-m}{2}) > \max(\gamma(x), \gamma(y)) = t$ for some $x, y \in G$, then $t \in [\frac{1-m}{2}, 1)$, $\gamma(x) > t$, $x \in L(\gamma; t)$ and $y \in L(\gamma; t)$. Since $x, y \in L(\gamma; t)$, $L(\gamma; t)$ is a subalgebra of $\mathcal{K}$, so $x \odot y \in L(\gamma; t)$, a contradiction. The proof of the sufficiency part is straightforward and is hence omitted. This completes the proof. □

Taking $m = 0$ in Theorem 30, we obtain the following corollary.

Corollary 31 Let γ be a fuzzy set of a K-algebra $\mathcal{K}$. Then the non-empty level set $L(\gamma; t)$ is a subalgebra of $\mathcal{K}$ for all $t \in [0.5, 1)$ if and only if γ satisfies the following assertion for all $x, y \in G$:

$$\min(\gamma(x \odot y), 0.5) \leq \max(\gamma(x), \gamma(y).$$

Theorem 32 Let γ be an $(\prec, \prec \vee\Upsilon_m)^*$-fuzzy subalgebra of $\mathcal{K}$.

(i) If there exists $x \in G$ such that $\gamma(x) \leq \frac{1-m}{2}$, then $\gamma(e) \leq \frac{1-m}{2}$.

(ii) If $\gamma(e) > \frac{1-m}{2}$, then γ is an anti fuzzy subalgebra of $\mathcal{K}$.

Proof:

(i) Let $x \in G$ such that $\gamma(x) \leq \frac{1-m}{2}$.

$$\gamma(x^{-1}) = \max(\gamma(x), \frac{1-m}{2}) = \frac{1-m}{2}.$$

Hence

$$\begin{aligned} \gamma(e) &= \gamma(x \odot x)) = \gamma(x.x^{-1}) \\ &\leq \max(\gamma(x), \gamma(x^{-1}), \frac{1-m}{2}) = \frac{1-m}{2}. \end{aligned}$$

(ii) $\gamma(e) > \frac{1-m}{2} \Rightarrow \gamma(x) > \frac{1-m}{2}$ for all $x \in G$. Thus we conclude that:

$$\gamma(x \odot y) \leq \max(\gamma(x), \gamma(y)), \forall\, x,\ y\ \in G.$$

Hence γ is an anti fuzzy subalgebra of $\mathcal{K}$.

□

Taking $m = 0$ in Theorem 32, we obtain the following corollary.

Corollary 33 Let γ be an $(\prec, \prec \vee \Upsilon)^*$-fuzzy subalgebra of $\mathcal{K}$.

(i) If there exists $x \in G$ such that $\gamma(x) \leq 0.5$, then $\gamma(e) \leq 0.5$.

(ii) If $\gamma(e) > 0.5$, then γ is an anti fuzzy subalgebra of $\mathcal{K}$.

CONCLUSIONS

In handling information regarding various aspects of uncertainty, non-classical logic (a great extension and development of classical logic) is considered to be more powerful technique than the classical logic one. The non-classical logic, therefore, has nowadays become a useful tool in computer science. Moreover, non-classical logic deals with the fuzzy information and uncertainty. In this connection we have introduced a new fuzzy subalgebra and have presented some properties of fuzzy subalgebra of a K-algebra using the notion of an anti fuzzy point and its besideness to and non-quasi-coincidence with a fuzzy set. In our opinion the future study of fuzzy K-algebras can be connected with: investigating (α, β)- bifuzzy K-ideals. Our obtained results can probably be applied in: (1) engineering; (2) computer science: artificial intelligence, signal processing; (3) medical diagnosis.

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