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A Minimax Inequality and its Applications to Fixed Point Theorems in R-trees

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Abstract: Recently, Kirk and Panyanak proved the KKM mapping principle in R-trees. In this paper, using this principle, we establish an R-tree version of famous Fan's minimax inequality and we apply this result to get some fixed point and best approximation theorems.

Key words: R-trees . minimax inequality . fixed point theorems . best approx-imation . KKM theorems

INTRODUCTION

R-trees, also known as metric trees, were first introduced by Tits [1]. The idea has been also studied in [2] and was called T-theory. An R-tree is a geodesic space for which there is a unique arc joining any two of its points and this arc is a metric segment details [3-5]. Many applications have been found for R-trees within different fields of mathematics [6]. Moreover there are applications in biology and computer science as well [7, 8].

The famous Knaster-Kuratowski-Mazurkiewicz theorem (in short, KKM theorem) [9] and its generalizations have a fundamental importance in modem nonlinear analysis. Recently, Kirk and Panyanak [5] established the KKM mapping principle for R-trees and applied it to prove an extension of Ky Fan's best approximation theorem to upper semicontinuous mappings in this setting. In this paper, a minimax inequality in R-trees is established and as an application of it, some fixed point and best approximation theorems in an R-tree setting are proved.

At first some relevant notations and terminologies are described. Let X be a metric space and $A\subset X$, henceforth we use Int (A), Bd (A), B (x,r) and conv_X (A) respectively to denote the interior of A, boundary of A, closed ball centered at x with radius $r\geq 0$ and the intersection of all closed convex subsets of X that contain A.

Let X and Y be two topological Hausdorff spaces and T: $X\to Y$ be a multivalued function with nonempty values if

$$T^{-1}(B):=\{x\in X: T(x)\cap B\neq\phi\}$$

then T is said to be:

- Upper semicontinuous, if for each closed set $B\subset Y$, $T^{-1}(B)\subset Y$ is closed in X
- Lower semicontinuous, if for each open set $B\subset Y$, $T^{-1}(B)$ is open in X
- Continuous, if it is both upper and lower semicontinuous.

Definition 1: An R-tree is a metric space X such that:

- There is a unique geodesic segment denoted by [x,y] joining each pair of points $x,y\in X$.
- If $[y,x]\bigcap[x,z]=\{x\}$, then $[y,x]\bigcup[x,z]=[y,z]$.
- If $x,y,z\in X$ then $[x,y]\bigcap[x,z]=[x,w]$ for some $w\in X$.

This notion was introduced in [1]. Standard examples of R-trees include the radial and river metrics on $\Re^2$. Much more subtle examples exist; e.g. the real tree in [10].

Definition 2: Let X be an R-tree, C be a closed convex subset of X. A function f: $C\to\Re$ is said to be metrically quasi-concave (resp., metrically quasi-convex) if for each $\lambda\in\Re$, the set $\{x\in C: f(x)\geq\lambda\}$ (resp., $\{x\in C: f(x)\leq\lambda\}$) is closed and convex.

Definition 3: Let C be a nonempty subset of an R-tree X. A multivalued mapping $G:C\to 2^X$ is said to be a KKM mapping if for each nonempty finite set $F\subset C$,

$$\mathrm{conv}_X(F)\subseteq\bigcup_{x\in F}G(x)$$

This notion was introduced in [5], in which the following important result is established.

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Theorem 1. (KKM mapping principle): Suppose C is a closed convex subset of a complete R-tree X and $G:C\to 2^X$ has nonempty closed values. If G is a KKM mapping then $\{G(x)\}_{x\in C}$ has a finite intersection property. Moreover, if C is geodesically bounded, then

$$\bigcap_{x\in C} G(x)\neq \phi.$$

MAIN RESULTS

We first establish the following lemma in order to prove our main results.

Lemma 1: Let C be a closed convex subset of an R-tree X and suppose $f: C\times C\to \Re$ satisfies:

- For each $y\in C$, the function $f(.,y): C\to\Re$ is metrically quasi-concave (resp., metrically quasi-convex),
- There exists $\gamma\in\Re$ such that $f(x,x)\le\gamma$ (resp., $f(x,x)\ge \gamma$) for each $x\in C$,

Then the mapping $G:C\to 2^X$ which is defined by:

$$G(x)=\{y\in C: f(x,y)\le\gamma\}(\text{resp.}, G(x)=\{y\in C: f(x,y)\ge\gamma\})$$

is a KKM mapping.

Proof: The conclusion is proved for the concave case; the convex case is completely similar. On the contrary assume that G is not a KKM mapping. Suppose that there exists a finite subset $F=\{x_1,\ldots,x_n\}$ of C and a point $x_0\in conv_X(F)$ such that $x_0\notin G(x_i)$ for each $i = 1,\ldots,n$. Then we have $f(x_i,x_0)>\gamma$ for each i. By setting

$$\lambda=\min\{f(x_i,x_0): i=1,\ldots,n\}>\gamma$$

and

$$A=\{z\in C: f(z,x_0)\ge\lambda\}$$

for each i, $x_i\in A$. According to hypothesis 1, A is closed and convex and hence $x_0\in conv_X(F)\subset A$. Therefore $f(x_0,x_0)\ge\lambda>\gamma$, this is a contradiction to hypothesis 2. Thus G is a KKM mapping.

The following is an R-tree version of the Fan's minimax inequality [11].

Theorem 1: Suppose C is a closed convex geodesically bounded subset of a complete R-tree X and $f: C\times C\to\Re$ satisfies:

- For each $x\in C$, the function $f(x,.):C\to\Re$ is lower semicontinuous (resp., upper semicontinuous);
- For each $y\in C$, the function $f(.,y):C\to\Re$ is metrically quasi-concave (resp., metrically quasi-convex);
- There exists $\gamma\in\Re$ such that $f(x,x)\le\gamma$ (resp., $f(x,x)\ge\gamma$) for each $x\in C$.

Then there exists a $y_0\in C$ such that $f(x,y_0)\le\gamma$ (resp., $f(x,y_0)\ge\gamma$) for all $x\in C$ and hence

$$\sup_{x\in C} f(x,y_0)\le \sup_{x\in C} f(x,x)$$
$$(\text{resp., } \inf_{x\in C} f(x,y_0)\ge \inf_{x\in C} f(x,x)).$$

Proof: By hypothesis 3, $\lambda=\sup_{x\in C} f(x,x)<\infty$. For each $x\in C$, define the mapping $G:C\to 2^X$ by:

$$G(x)=\{y\in C: f(x,y)\le\lambda\}.$$

which is closed by hypothesis 1. By Lemma 1, G is a KKM mapping. Since C is a closed convex and geodesically bounded, by using of the KKM mapping principle, $\bigcap_{x\in C} G(x)\neq\phi$

Therefore there exists a y_0 in this intersection. Thus $f(x,y_0)\le\lambda$ for all $x\in C$ and hence

$$\sup_{x\in C} f(x,y_0)\le \sup_{x\in C} f(x,x)$$

Now as an application of minimax inequality in R-trees, we prove some fixed point and best approximation theorems. The following result is an extension of Fan's best approximation theorem to lower semicontinuous multivalued mappings in an R-tree setting. Note that we use $c(X)$ to denote all closed convex subsets of X.

Theorem 2: Suppose X is a complete R-tree, C is a closed convex geodesically bounded subset of X and $T:C\to c(X)$ is lower semicontinuous. Then there exists a $y_0\in C$ such that

$$d(y_0, T(y_0))=\inf_{x\in C} d(x,T(y_0)).$$

Moreover, if $y_0\notin T(y_0)$ then $y_0\in Bd(C)$.

Proof: Define the mapping $f:C\times C\to\Re$ by

$$f(x,y)=d(y,Ty)-d(x,Ty),\quad \forall x,y\in C.$$

It is proved that f satisfies the hypotheses in Theorem 1. Obviously, $f(x,.)$ is lower semicontinuous. For each $y\in C$ and $\lambda\in\Re$, let

$$A=\{x\in C: d(y,Ty)-d(x,Ty)\ge \lambda\}=\{x\in C: d(x,Ty)\le\gamma\},$$

Where $\gamma = d(y,Ty) - \lambda$.

Since C is a subtree of X and Ty is closed and convex, A is also closed and convex. Hence f(.,y) is metrically quasi-concave.

Obviously *f*(x,x) = 0 for all $x \in C$. Therefore by Theorem 1, there exists $y_0 \in C$ such that

$$d(y_0, Ty_0) \le d(x, Ty_0), \qquad \forall x \in C \qquad \text{(i)}$$

which implies that

$$d(y_0, T(y_0)) = \inf_{x \in C} d(x, T(y_0)).$$

Suppose $y_0 \notin T(y_0)$ and $d(y_0, T(y_0)) = r > 0$. We prove that $y_0 \notin Bd(C)$. On the contrary assume that $y_0 \notin Int(C)$; then there exists an $\varepsilon \in (0,r)$ such that $B(y_0, \varepsilon) \subset C$. Take $z_0 \in Ty_0$ such that $d(y_0, z_0) < r + \varepsilon/2$. By metric convexity of X, take $x_0 \in [y_0, z_0]$

Such that $d(y_0, x_0) = \varepsilon/2$. Since X is an R-tree, we have

$$\begin{aligned} d(x_0, Ty_0) \le d(x_0, z_0) = d(y_0, z_0) - d(y_0, x_0) < r \\ = d(y_0, Ty_0), \end{aligned}$$

which contradicts (i). Therefore $y_0 \in Bd(C)$.

Corollary 1: Let X,C,T be the same as Theorem 2. If $Tx \cap C \neq \phi$ for all $x \in Bd(C)$, then T has a fixed point.

Proof: On the contrary assume that T doesn't have a fixed point. Therefore by Theorem 2 there exists an $y_0 \in Bd(C)$ such that

$$0 < d(y_0, Ty_0) \le d(x, Ty_0), \qquad \forall x \in C \qquad \text{(ii)}$$

Since $y_0 \in Bd(C)$, we have $Ty_0 \cap C \neq \phi$. Thus by (ii) we get $d(y_0, Ty_0) = 0$, which is a contradiction.

If in Theorem 2, T is single valued then it reduces to the following analog of Fan's best approximation theorem to point valued mappings in geodesically bounded R-trees which was proved in [4] (Theorem 3.6.).

Corollary 2: Suppose X is a complete R-tree, C is a closed convex geodesically bounded subset of X and $T: C \to X$ is continuous. Then there exists an $y_0 \in C$ such that

$$d(y_0, Ty_0) \le d(x, Ty_0), \qquad \forall x \in C$$

The following is an analog of Fan's fixed point theorem [12] in R-trees.

Theorem 3: Suppose X is a complete R-tree, C is a closed convex geodesically bounded subset of X and $T: C \to X$ is continuous and for every $c \in X$ with $c \neq T(c)$,

$$(c, T(c)] := [c, T(c)] \setminus \{c\}$$

contains at least one point of C, then T has a fixed point.

Proof: By the Corollary 2, there exists $y_0 \in C$ such that

$$d(y_0, Ty_0) \le d(x, Ty_0), \qquad \forall x \in C \qquad \text{(iii)}$$

We claim that y_0 is a fixed point of T. On the contrary assume that $y_0 \neq Ty_0$. Then by assumption there exists a $z \in C$ such that $z \in (y_0, Ty_0]$. Therefore

$$d(z, Ty_0) = d(y_0, Ty_0) - d(y_0, z) < d(y_0, Ty_0),$$

which is a contradiction to (iii).

Note that if $T(C) \subset C$, then the hypothesis of Theorem 3 is satisfied. Therefore as an immediate consequence, we obtain the following fixed point theorem which is due to Kirk ([4], Theorem 3.4.), by different method; however it is also a direct consequence of Theorem 2.

Corollary 3: Suppose X is a geodesically bounded complete R-tree. Then every continuous mapping $T: X \to X$ has a fixed point.

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