

Modified Variation of Parameters Method for Second-order Integro-differential Equations and Coupled Systems

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Abstract: In this paper, we apply the Modified Variation of Parameters Method (MVPM) for solving second-order integro-differential equations and coupled systems of integro-differential-equations. The proposed modification is made by the elegant coupling of He's polynomials and the Ma's variation of parameters method. The proposed MVPM is applied without any discretization, transformation or restrictive assumptions and is free from round off errors, calculation of the so-called Adomian's polynomials and the identification of Lagrange multipliers. The numerical results are very encouraging.

Key words: Variation of parameters method . he's polynomials . integro differential-equations . blasius problem . error estimates

INTRODUCTION

The integro-differential equations are of great significance and are the governing equations of diversified nonlinear physical problems related to physics, astrophysics, magnetic dynamics, water surface, gravity waves, ion acoustic waves in plasma, electromagnetic radiation reactions, engineering and applied sciences [1-21]. Several techniques including decomposition, homotopy perturbation, polynomial and non polynomial spline, Sink Galerkin, perturbation, homotopy analysis, finite difference and modified variational iteration have been employed to solve such problems [1-21] and the reference therein. Most of these used schemes are coupled with the inbuilt deficiencies like calculation of the so-called Adomian's polynomials, identification of Lagrange multiplier and non compatibility with the physical nature of the problems. These facts motivated to suggest alternate techniques such as the Variation of Parameters Method (VPM) [8-10]. Ma and You [8-10] used variation of parameters for solving involved non-homogeneous partial differential equations and obtained solution formulas helpful in constructing the existing solutions coupled with a number of other new solutions including rational solutions, solitons, positions, negatons, breathers, complexions and interaction solutions of the KdV equations. Recently, Mohyud-Din, Noor and Noor [15-17] developed the Modified Variation of Parameters Method (MVPM) which is obtained by the elegant coupling of He's polynomials and the variation of parameters method. The basic motivation of this

paper is the implementation of Modified Variation of Parameters Method (MVPM) for solving for solving second-order integro-differential equations and coupled system of integro-differential equations.. The Modified Variation of Parameter Method (MVPM) is applied without any discretization, transformation or restrictive assumptions. The suggested method is free from round off errors and calculation of the so-called Adomian's polynomials. The numerical results are very encouraging.

VARIATION OF PARAMETERS METHOD (VPM)

Consider the following second-order partial differential equation

$$y_{tt} = f(t, x, y, z, y_x, y_y, y_z, y_{xx}, y_{yy}, y_{zz}) \quad (1)$$

where t such that $(-\infty < t < \infty)$ is time and f is linear or non linear function of $y, y_x, y_y, y_z, y_{xx}, y_{yy}, y_{zz}$.

The homogeneous solution of (1) is :

$$y(t, x, y, z) = A + Bt$$

where A and B are functions of x, y, z and t. Using Variation of parameters method we have following system of equations

$$\frac{\partial A}{\partial t} + \frac{\partial B}{\partial t} = 0$$

$$\frac{\partial B}{\partial t} = f$$

and hence

$$A(x,y,z,t) = D(x,y,z) - \int_0^t sfds$$

$$B(x,y,z,t) = C(x,y,z) - \int_0^t f ds$$

therefore,

$$y(x,y,z,t) = y(x,y,z,0) + ty_t(x,y,z,0)$$

$$+ \int_0^t (t-s)f(s,x,y,z,y_x, y_{yy}, y_{xx}, y_{zz})ds$$

which can be solved iteratively as [8-10, 15-17]

$$y^{k+1}(x,y,z,t) = y(x,y,z,0) + ty_t(x,y,z,0)$$

$$+ \int_0^t (t-s)f(s,x,y,z,y_x^k, y_y^k, y_z^k, y_{xx}^k, y_{yy}^k, y_{zz}^k)ds$$

$$k = 0, 1, 2, \dots$$

HOMOTOPY PERTURBATION METHOD (HPM) AND HE'S POLYNOMIALS

To explain the He's homotopy perturbation method, we consider a general equation of the type,

$$L(u) = 0 \quad (2)$$

where L is any integral or differential operator. We define a convex homotopy $H(u, p)$ by

$$H(u, p) = (1-p)F(u) + pL(u) \quad (3)$$

where $F(u)$ is a functional operator with known solutions v_0 , which can be obtained easily. It is clear that, for

$$H(u, p) = 0 \quad (4)$$

we have

$$H(u, 0) = F(u)$$

$$H(u, 1) = L(u)$$

This shows that $H(u, p)$ continuously traces an implicitly defined curve from a starting point $H(v_0, 0)$ to a solution function $H(f, 1)$. The embedding parameter monotonically increases from zero to unit as the trivial problem $F(u) = 0$ is continuously deforms the original problem $L(u) = 0$. The embedding parameter $p \in (0, 1]$ can be considered as an expanding parameter [3-7, 11-21]. The homotopy perturbation method uses the homotopy parameter p as an expanding parameter [3-7] to obtain

$$u = \sum_{i=0}^{\infty} p^i u_i = u_0 + p u_1 + p^2 u_2 + p^3 u_3 + \dots \quad (5)$$

if $p \rightarrow 1$, then (5) corresponds to (3) and becomes the approximate solution of the form,

$$f = \lim_{p \rightarrow 1} u = \sum_{i=0}^{\infty} u_i \quad (6)$$

It is well known that series (6) is convergent for most of the cases and also the rate of convergence is dependent on $L(u)$; [3-7]. We assume that (6) has a unique solution. The comparisons of like powers of p give solutions of various orders. In sum, according to [3], He's HPM considers the nonlinear term $N(u)$ as:

$$N(u) = \sum_{i=0}^{\infty} p^i H_i = H_0 + p H_1 + p^2 H_2 + \dots$$

where H_n 's are the so-called He's polynomials [3], which can be calculated by using the formula

$$H_n(u_0, \dots, u_n) = \frac{1}{n!} \frac{\partial^n}{\partial p^n} \left(N \left(\sum_{i=0}^n p^i u_i \right) \right)_{p=0}, \quad n = 0, 1, 2, \dots$$

NUMERICAL APPLICATIONS

In this section, we apply the Modified Variation of Parameters Method (MVPM) for solving second-order integro-differential equations and coupled system of integro-differential equations. Numerical results are very encouraging.

Example 4.1: Consider the following second-order system of nonlinear integro-differential equations

$$u''(x) = 1 - \frac{1}{3}x^3 - \frac{1}{2}(v')^2 + \frac{1}{2} \int_0^x (u^2(t) + v^2(t))dt$$

$$v''(x) = -1 + x^2 - xu(x) + \frac{1}{4} \int_0^x (u^2(t) - v^2(t))dt$$

with initial conditions

$$u(0) = 1, \quad u'(0) = 2, \quad v(0) = -1, \quad v'(0) = 0$$

The exact solutions for this problem is

$$u(x) = x + e^x, \quad v(x) = x - e^x$$

Applying Variation of Parameters Method (VPM)

Table 1: Error estimates

x	*Errors	
	U	V
-1.0	2.16E-03	8.74E-04
-0.8	4.80E-04	1.85E-04
-0.6	6.87E-05	2.49E-05
-0.4	4.35E-06	1.46E-06
-0.2	3.71E-08	1.15E-08
0.0	0.00000	0.00000
0.2	4.50E-08	1.10E-08
0.4	6.27E-06	1.42E-06
0.6	1.17E-04	2.38E-05
0.8	9.69E-04	1.73E-04
1.0	5.06E-03	7.89E-04

Error = Exact solution-series solution

$$\begin{cases} u_{n+1}(x) = u_n(x) + \int_0^x (x-s) \left(\left(1 - \frac{1}{3}x^3 - \frac{1}{2}(v_n')^2 \right) + \frac{1}{2} \int_0^x (u_n^2(t) + v_n^2(t)) dt \right) ds \\ v_{n+1}(x) = v_n(x) + \int_0^x (x-s) \left((-1+x^2 - x u_n(x)) + \frac{1}{2} \int_0^x (u_n^2(t) - v_n^2(t)) dt \right) ds \end{cases}$$

Applying the Modified Variation of Parameters Method (MVPBM)

$$\begin{cases} u_0 + pu_1 + \dots = 1 + 2x + p \int_0^x (x-s) \left(\left(1 - \frac{1}{3}x^3 - \frac{1}{2}(v_0' + pv_1' + \dots)^2 \right) + \frac{1}{2} \int_0^x ((u_0^2 + pu_1^2 + \dots) + (v_0^2 + pv_1^2 + \dots)) ds \right) ds \\ v_0 + pv_1 + \dots = -1 + p \int_0^x (x-s) \left((-1+x^2 - x(u_0 + pu_1 + \dots)) + \frac{1}{2} \int_0^x ((u_0^2 + pu_1^2 + \dots) - (v_0^2 + pv_1^2 + \dots)) ds \right) ds \end{cases}$$

Comparing co-efficient of like powers of p, following approximations are obtained

$$\begin{aligned} p^{(0)} : & \begin{cases} u_0(x) = 1 + 2x \\ v_0(x) = -1 \end{cases} \\ p^{(1)} : & \begin{cases} u_1(x) = \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{60}x^5 \\ v_1(x) = -\frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{24}x^4 + \frac{1}{60}x^5 \end{cases} \\ p^{(2)} : & \begin{cases} u_2(x) = -\frac{1}{24}x^4 - \frac{1}{120}x^5 + \frac{1}{720}x^6 + \frac{17}{5040}x^7 + \frac{1}{672}x^8 + \frac{53}{120960}x^9 - \frac{1}{103680}x^{10} + \frac{17}{228096}x^{11} \\ \quad + \frac{1}{1900800}x^{12} + \frac{1}{6177600}x^{13} \\ v_2(x) = \frac{1}{120}x^5 - \frac{1}{720}x^6 - \frac{11}{10080}x^7 + \frac{13}{241920}x^9 + \frac{17}{1036800}x^{10} + \frac{47}{11404800}x^{11} + \frac{1}{1267200}x^{12} \end{cases} \end{aligned}$$

The series solution is given by

$$\begin{cases} u(x) = 1 + 2x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{60}x^5 - \frac{1}{24}x^6 - \frac{1}{120}x^7 + \frac{1}{720}x^8 + \frac{17}{5040}x^9 + \frac{1}{672}x^{10} + \frac{53}{120960}x^{11} \\ \quad - \frac{1}{103680}x^{12} + \frac{17}{228096}x^{13} + \frac{1}{1900800}x^{14} + \frac{1}{6177600}x^{15} + \dots \\ v(x) = -1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{24}x^4 + \frac{1}{60}x^5 - \frac{1}{120}x^6 - \frac{1}{720}x^7 - \frac{11}{10080}x^8 + \frac{13}{241920}x^9 + \frac{17}{1036800}x^{10} + \frac{47}{11404800}x^{11} + \frac{1}{1267200}x^{12} + \dots \end{cases}$$

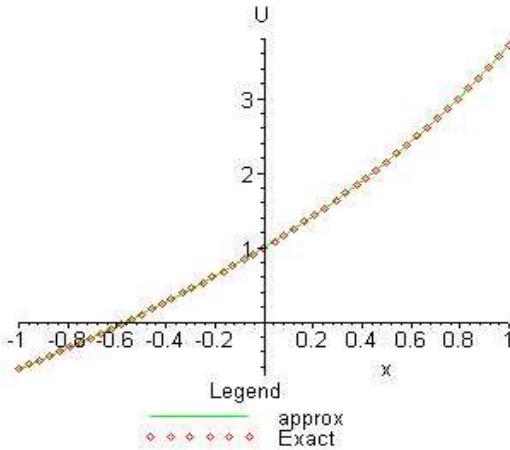


Fig. 1:

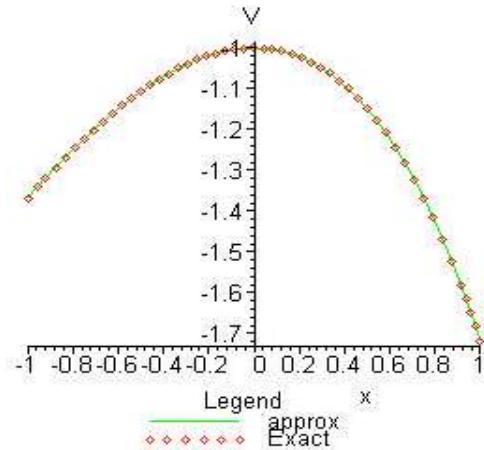


Fig. 2:

Example 4.2: Consider the following second-order system of nonlinear integro-differential equations

$$\begin{aligned} u''(x) &= x + 2x^3 + 2(v'(x))^2 - \int_0^x ((v'(t))^2 + u(t)w'(t)v^2(t))dt \\ v''(x) &= 3x^2 - xu(x) + \frac{1}{4} \int_0^x (txv'(t)u''(t) - w'(t))dt \\ w''(x) &= 2 - \frac{4}{3}x^3 + (u''(x))^2 - 2u^2(x) + \int_0^x x(v^2(t) + (u'(t))^2 + t^3w''(t))dt \end{aligned}$$

with initial conditions

$$u(0)=1, \quad u'(0)=0, \quad v(0)=0, \quad v'(0)=1, \quad w(0)=0, \quad w'(0)=0$$

The exact solutions for this problem is

$$u(x) = x^2, \quad v(x) = x, \quad w(x) = 3x^2$$

Applying Variation of Parameters Method (VPM)

$$\begin{cases} u_{n+1}(x) = u_n(x) + \int_0^x (x-s) \left(\left(x + 2x^3 + 2(v'_n(x))^2 \right) - \int_0^s ((v'_n(t))^2 + u_n(t)w''_n(t)v^2(t)) dt \right) ds \\ v_{n+1}(x) = v_n(x) + \int_0^x (x-s) \left(\left(3x^2 - xu_n(x) \right) + \frac{1}{4} \int_0^s ((txv'_n(t)u''_n(t) - w'_n(t))) dt \right) ds \\ w_{n+1}(x) = w_n(x) + \int_0^x (x-s) \left(\left(2 - \frac{4}{3}x^3 + (u''_n(x))^2 - 2u_n^2(x) \right) - \int_0^s x(v_n^2(t) + (u'_n(t))^2 + t^3w''_n(t)) dt \right) ds \end{cases}$$

Applying the Modified Variation of Parameters Method (MVPM)

$$\begin{cases} u_0 + pu_1 + \dots = 1 + p \int_0^x (x-s) \left(\left(x + 2x^3 + 2(v'_0 + pv'_1 + \dots)^2 \right) + \int_0^s (v'_0 + pv'_1 + \dots)^2 ds \right) - \int_0^x (x-s) \left((u_0 + pu_1 + \dots)(v_0 + pv_1 + \dots)^2 (w''_0 + pw''_1 + \dots) \right) ds \\ v_0 + pv_1 + \dots = x + p \int_0^x (x-s) \left(\left(3x^2 - x(u_0 + pu_1 + \dots) \right) + \frac{1}{4} \int_0^s (sx(v'_0 + pv'_1 + \dots)(u_0^2 + pu_1^2 + \dots)(w''_0 + pw''_1 + \dots)) ds \right) + p \int_0^x (x-s) \left((v'_0 + pv'_1 + \dots)(u_0^2 + pu_1^2 + \dots)(w''_0 + pw''_1 + \dots) \right) ds \\ w_0 + pw_1 + \dots = p \int_0^x (x-s) \left(\left(2 - \frac{4}{3}x^3 + (u''_0 + pu''_1 + \dots)^2 - 2(u_0 + pu_1 + \dots)^2 \right) - \int_0^s x(v_0 + pv_1 + \dots)^2 ds \right) - p \int_0^x x(v_0 + pv_1 + \dots)^2 ds - p \int_0^x x(x-s) \left((u'_0 + pu'_1 + \dots)^2 + t^3(w''_0 + pw''_1 + \dots) \right) ds \end{cases}$$

Comparing co-efficient of like powers of p, following approximations are obtained

Table 2: Error estimates

x	*Errors		
	U	V	W
-1.0	2.33E-02	3.48E-02	3.48E-01
-0.8	4.79E-03	1.25E-03	1.25E-01
-0.6	5.77E-04	3.12E-04	3.12E-02
-0.4	2.83E-05	4.17E-05	4.17E-03
-0.2	1.72E-07	1.29E-07	1.29E-04
0.0	0.00000	0.00000	0.00000
0.2	7.20E-08	1.26E-08	1.26E-04
0.4	2.80E-06	4.00E-06	4.00E-03
0.6	7.15E-05	3.07E-05	3.07E-02
0.8	1.64E-03	1.34E-03	1.34E-01
1.0	1.53E-02	4.37E-02	4.37E-01

Error = Exact solution-series solution

$$\begin{aligned}
 p^{(0)}: & \begin{cases} u_0(x) = 1 \\ v_0(x) = x \\ w_0(x) = 0 \end{cases} \\
 p^{(1)}: & \begin{cases} u_1(x) = x^2 + \frac{1}{10}x^5 \\ v_1(x) = \frac{1}{4}x^4 \\ w_1(x) = x^2 - \frac{1}{15}x^5 + \frac{1}{60}x^6 \end{cases} \\
 p^{(2)}: & \begin{cases} u_2(x) = \frac{1}{6}x^5 - \frac{1}{60}x^6 + \frac{197}{5040}x^8 \\ \quad - \frac{1}{336}x^9 + \frac{1}{7425}x^{11} - \frac{17}{26400}x^{12} \\ v_2(x) = -\frac{1}{12}x^4 - \frac{1}{630}x^7 - \frac{41}{3360}x^8 + \frac{1}{440}x^{11} \\ w_2(x) = -\frac{1}{20}x^6 + \frac{13}{168}x^8 - \frac{227}{30240}x^9 \\ \quad + \frac{1}{1440}x^{10} + \frac{1}{3960}x^{11} - \frac{1}{6600}x^{12} \\ \vdots \end{cases}
 \end{aligned}$$

The series solution is given by

$$\begin{cases} u(x) = 1 + x^2 + \frac{1}{10}x^5 + \frac{1}{6}x^5 - \frac{1}{60}x^6 + \frac{197}{5040}x^8 \\ \quad - \frac{1}{336}x^9 + \frac{1}{7425}x^{11} - \frac{17}{26400}x^{12} + \dots \\ v(x) = x + \frac{1}{4}x^4 - \frac{1}{12}x^4 - \frac{1}{630}x^7 - \frac{41}{3360}x^8 + \frac{1}{440}x^{11} + \dots \\ w(x) = x^2 - \frac{1}{15}x^5 + \frac{1}{60}x^6 - \frac{1}{20}x^6 + \frac{13}{168}x^8 - \frac{227}{30240}x^9 \\ \quad + \frac{1}{1440}x^{10} + \frac{1}{3960}x^{11} - \frac{1}{6600}x^{12} + \dots \end{cases}$$

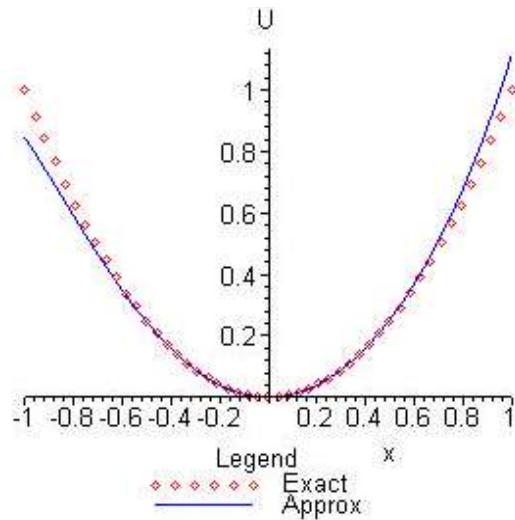


Fig. 3:

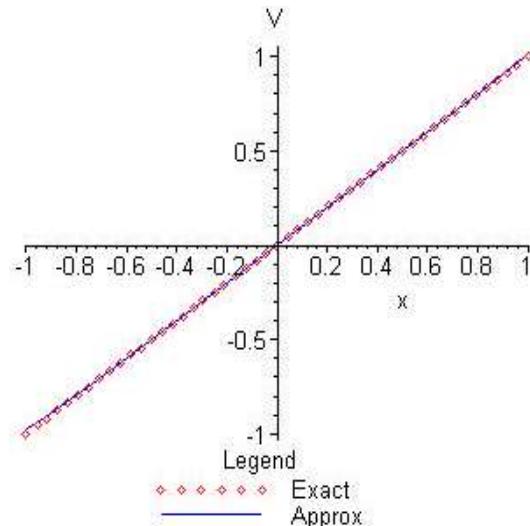


Fig. 4:

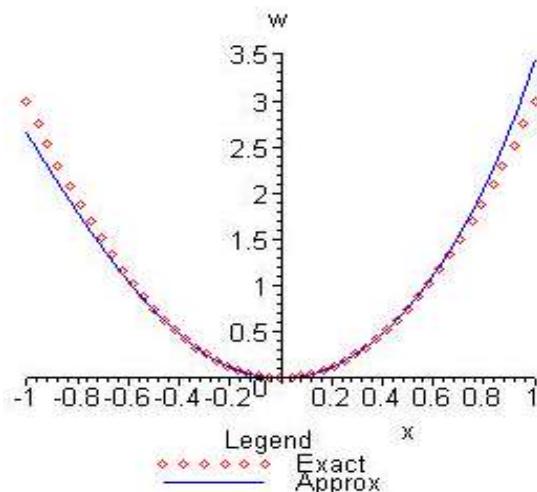


Fig. 5:

Table 3: Pade' approximants and numerical value of α

Pade' approximant	α
[2/2]	0.5778502691
[3/3]	0.5163977793
[4/4]	0.5227030798

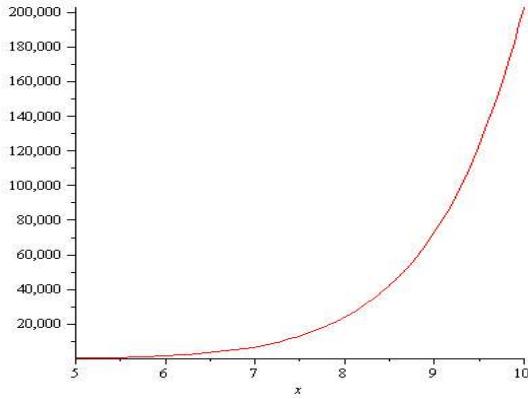


Fig. 6: $\alpha = 0.5778502691$

Example 4.3: Consider the two dimensional nonlinear inhomogeneous initial boundary value problem for the integro differential equation related to the Blasius problem

$$y''(x) = \alpha - \frac{1}{2} \int_0^x y(t)y''(t)dt, \quad -\infty < x < 0$$

with boundary conditions

$$y(0) = 0, \quad y'(0) = 1$$

and

$$\lim_{x \rightarrow \infty} y'(x) = 0$$

Applying the Modified Variation of Parameters Method (MVP)

$$y_0 + py_1 + p^2y_2 + \dots = x + p \int_0^x (x-s) \left(\int_0^x (y_0 + py_1 + \dots)(y''_0 + py''_1 + p^2y''_2 + \dots) ds \right) ds$$

Proceeding as before, the series solution is given as

$$\begin{aligned} y(x) = & x + \frac{1}{2} \alpha x^2 - \frac{1}{48} \alpha x^4 - \frac{1}{240} \alpha^2 x^5 + \frac{1}{960} \alpha x^6 + \frac{11}{20160} \alpha^2 x^7 + \left(\frac{11}{161280} \alpha^3 + \frac{1}{960} \alpha \right) x^8 - \frac{43}{967680} \alpha^2 x^9 \\ & + \left(\frac{1}{52960} \alpha - \frac{5}{387072} \alpha^3 \right) x^{10} + \left(\frac{587}{21288960} \alpha^2 - \frac{5}{4257792} \alpha^4 \right) x^{11} + \left(-\frac{1}{16220160} \alpha + \frac{1}{7257792} \alpha^3 \right) x^{12} + \dots \end{aligned}$$

and consequently

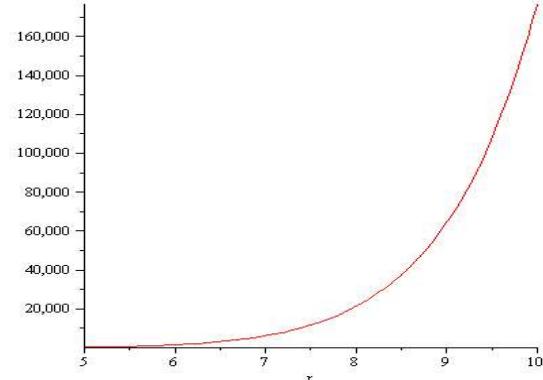


Fig. 7: $\alpha = 0.5163977793$

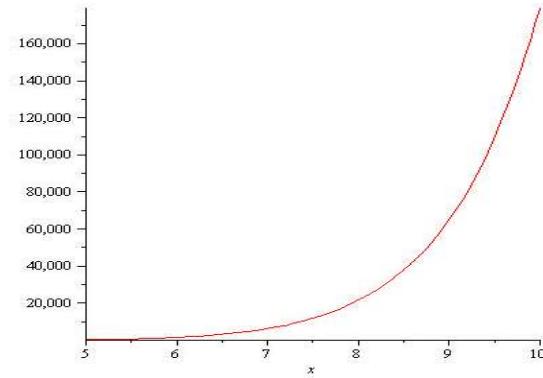


Fig. 8: $\alpha = 0.5227030798$

where constant α is positive and defined by

$$y'(0) = \alpha \quad \alpha > 0$$

Applying the Variation of Parameters Method (VPM)

$$y_{n+1}(x) = x + \int_0^x (x-s) \left(\alpha - \frac{1}{2} \int_0^s (y_n(s)y''_n(s)) ds \right) ds \quad -\infty < x < 0$$

$$y'(x) = 1 + \alpha x - \frac{1}{12} \alpha x^3 - \frac{1}{48} \alpha^2 x^4 + \frac{1}{160} \alpha x^5 + \frac{11}{2880} \alpha^2 x^6 \left(\frac{11}{20160} \alpha^3 - \frac{1}{2688} \alpha \right) x^7 - \frac{43}{107520} \alpha^2 x^8 \\ + 10 \left(\frac{1}{552960} \alpha - \frac{5}{387072} \alpha^3 \right) x^9 + 11 \left(\frac{587}{212889600} \alpha^2 - \frac{5}{4257792} \alpha^4 \right) x^{10} + 12 \left(-\frac{1}{16220160} \alpha + \frac{1}{725760} \alpha^3 \right) x^{11} + \dots$$

The diagonal Padé' approximants can be applied to determine a numerical value for the constant α by using the given condition.

CONCLUSION

In this paper, we applied Modified Variation of Parameters Method (MVPM) for solving integro-differential equations and coupled systems of integro-differential equations. The proposed technique is employed without using linearization, discretization or restrictive assumptions. Moreover, the suggested method is free from round off errors, calculation of the so-called Adomian's polynomials and identification of Lagrange multiplier. It may be concluded that the MVPM is very powerful and efficient in finding the analytical solutions for a wide class of boundary value problems and can be considered as an alternative for solving nonlinear initial and boundary value problems.

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REFERENCES

- Abbasbandy, 2006. The application of homotopy analysis method to nonlinear equations arising in heat transfer. *Phys. Lett.*, A360: 109-113.
- Abdou, A. and A.A. Soliman, 2005. Variational iteration method for solving Burger's and coupled Burger's equations. *J. Comput. Appl. Math.*, 181: 245-251.
- Ghorbani and J.S. Nadjfi, 2007. He's homotopy perturbation method for calculating Adomian's polynomials. *Int. J. Nonlin. Sci. Num. Simul.*, 8 (2): 229-332.
- He, H., 2008. An elementary introduction of recently developed asymptotic methods and nanomechanics in textile engineering. *Int. J. Mod. Phys.*, B22 (21): 3487-4578.
- He, H., 2008. Recent developments of the homotopy perturbation method. *Top. Meth. Nonlin. Anal.*, 31: 205-209.
- He, H., 2006. Some asymptotic methods for strongly nonlinear equation. *Int. J. Mod. Phys.*, 20 (10): 1144-1199.
- He, H., 2004. Comparison of homotopy perturbation method and homotopy analysis method. *Appl. Math. Comput.*, 156: 527-539.
- Ma, X. and Y. You, 2004. Solving the Korteweg-de Vries equation by its bilinear form: Wronskian solutions. *Transactions of the American mathematical society*, 357: 1753-1778.
- Ma, W.X. and Y. You, 2004. Rational solutions of the Toda lattice equation in Casoratian form. *Chaos, Solitons and Fractals*, 22: 395-406.
- Ma, W.X., C.X. Li and J.S. He, 2008. A second Wronskian formulation of the Boussinesq equation. *Nonlin. Anl. Th. Meth. Appl.*, 70 (12): 4245-4258.
- Mohyud-Din, S.T., M.A. Noor and K.I. Noor, 2009. Travelling wave solutions of seventh-order generalized KdV equations using He's polynomials. *Int. J. Nonlin. Sci. Num. Sim.*, 10 (2): 223-229.
- Mohyud-Din, S.T. and M.A. Noor, 2009. Homotopy perturbation method for solving partial differential equations. *Zeitschrift für Naturforschung A*, 64a.
- Mohyud-Din, S.T., M.A. Noor and K.I. Noor, 2009. Some relatively new techniques for nonlinear problems. *Math. Prob. Eng.*
- Mohyud-Din, S.T. and M.A. Noor, 2007. Homotopy perturbation method for solving fourth-order boundary value problems. *Math. Prob. Eng.*, Article ID 98602, doi:10.1155/2007/98602, pp: 1-15.
- Mohyud-Din, S.T., M.A. Noor and K.I. Noor, 2009. Modified variation of parameters method for solving partial differential equations. *Int. J. Mod. Phys. B*.
- Mohyud-Din, S.T., M.A. Noor, K.I. Noor and A. Waheed, 2009. Modified variation of parameters method for solving nonlinear boundary value problems. *Int. J. Mod. Phys. B*.

17. Mohyud-Din, S.T., M.A. Noor and K.I. Noor, 2009. Ma's variation of parameters method for nonlinear oscillator differential equations. *Int. J. Mod. Phys. B*.
18. Mohyud-Din, S.T., M.A. Noor and K.I. Noor, 2008. Comparison and coupling of polynomials with correction functional for travelling wave solutions of seventh order generalized KdV equations, *Wd. Appl. Sci. J.*
19. Mohyud-Din, S.T., M.A. Noor and K.I. Noor, 2009. Solving second-order singular problems using He's polynomials. *Wd. Appl. Sci. J.*, 6 (6): 769-775.
20. Noor, M.A. and S.T. Mohyud-Din, 2008. Modified variational iteration method for Goursat and Laplace problems. *Wd. Appl. Sci. J.*, 4 (4): 487-498.
21. Noor, M.A., S.T. Mohyud-Din and M. Tahir, 2008. Modified variational iteration methods for Thomas-Fermi equation. *Wd. Appl. Sci. J.*, 4 (4): 479-498.