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On the Analytical Solution of Kirchhoff Simplified Model for Beam by using of Homotopy Analysis Method

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Abstract: In this paper the problem of Kirchhoff simplified model for beam is studied and analytical solution is presented. Governing equation is solved analytically using a kind of analytic technique for nonlinear problems namely the Homotopy Analysis Method (HAM), for the first time. Present solution gives an expression which can be used in wide range of time for all domain of response and method is able to control the convergence of the solution. Comparisons of the obtained solutions with numerical results show that this method is effective and convenient for solving this problem. This method is a capable tool for solving this kind of nonlinear problems.

Key words: Analytical solution . HAM . Kirchhoff simplified beam . Convergence

INTRODUCTION

Most scientific problems in engineering are inherently nonlinear. Except a few number of them, majority of nonlinear problems do not have analytical solution. Therefore, these nonlinear equations should be solved using other methods Such as numerical or Perturbation method. In the numerical method, stability and convergence should be considered so as to avoid divergence or inappropriate results each effective parameter should be solved iteratively [1]. In the perturbation method, the small parameter is inserted in the equation. Thus, finding the small parameter and exerting it into the equation is one of the deficiencies of this method [2].

One of the semi-exact methods for solving nonlinear equation which does not need small/large parameters is Homotopy Analysis Method (HAM), first proposed by Liao in 1992 [3, 4]. This method is applied successfully to solve many problems in solid and fluid mechanics [5-11]. The convergence region can be adjusted and controlled which is effective feature of this technique in comparison to the other techniques.

For multi-degree of freedom or continuous nonlinear vibratory systems, how to analyze their qualitative and quantitative behavior is still a problem to be explored. As a possible approach, the concept and method of nonlinear normal mode (NNM) were proposed and studied by Rosenberg in 1960s and especially, attracted extensive studies since 1990 [12].

The concept of a normal mode is central in the theory of linear vibrating systems. Besides their obvious physical interpretation, the Linear Normal Modes (LNMs) have interesting mathematical properties. They can be used to decouple the governing equations of motion; i.e., a linear system vibrates as if it were made of independent oscillators governed by the eigensolutions.

In the other hand, clearly, though, linearity is an idealization, an exception to the rule; nonlinearity is a frequent occurrence in real-life applications [13, 14]. Thus, there is a need for efficient, analytically rigorous, broadly applicable analysis techniques for nonlinear structural dynamics. Nonlinear normal modes (NNMs) is a solid theoretical and mathematical tool for interpreting a wide class of nonlinear dynamical phenomena, yet they have a clear and simple conceptual relation to the LNMs, with which practicing structural engineers are familiar. Other appealing features of the NNMs are that they are capable of handling strong structural nonlinearity and that they have the potential to address the individualistic nature of nonlinear systems [15]. There are a few works on analytical solution for nonlinear beam, Wang *et al*. solved the cantilevered beam under nonconservative load by HAM [16] and Pirbodaghi *et al*. obtained the nonlinear vibration of bean [17].

First definition of nonlinear normal modes for continuous system (NNMCS) is based on exact separation of spatial and time variables. Exact separation of spatial and time variables is possible for some simplified models of continuous systems (Kirchhoff model of nonlinear beam [18], nonlinear sliding mode control design for shunt active power filter [19] Berger model of nonlinear plate [18], Berger-like model of shallow shell [20-22]).

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Up to now, no investigation has been made which provides the analytical solution for the problem of Kirchhoff simplified model for beam. In this study, HAM is applied to find an analytical solution of nonlinear ordinary differential equations arising from the nonlinear normal mode and the results were compared with the numerical solution based on shooting method and fourth order Runge Kutta method developed by these authors.

GOVERNING EQUATIONS

By using Kirchhoff simplified model for beam [18], we have:

$$\rho F\frac{\partial^2 W}{\partial t^2}+EI\frac{\partial^4 W}{\partial s^4}-\frac{EF}{2l}\int_0^l\left(\frac{\partial W}{\partial s}\right)^2 dx\frac{\partial^2 W}{\partial s^2}=0 \qquad (1)$$

where E is Young modulus, F and I are the area and second moment of area of the beam cross-section, respectively, W is normal displacement, ρ is mass per unit area, s is axial coordinate and t is time. It must be noted that Kirchhoff model must be used carefully because very often it is used in an improper way.
In this study, it can be assumed:

$$u=0 \qquad (\text{for } s=0,l) \qquad (2)$$

where u is axial displacement and l is the length of beam. Let us suppose the following boundary conditions for W:

$$W=0,\ \frac{\partial^2 W}{\partial s^2}-c\frac{\partial W}{\partial s}=0 \qquad (\text{for } s=0,l) \qquad (3)$$

where c = C/EI and C is end spring stiffness.

For boundary conditions (3) exact separation of time and space variables is impossible. We can use asymptotic procedure-dynamical edge effect approach, proposed for linear problems by Bolotin [23] and generalized for nonlinear case in Refs. [24, 25]. As small parameter quantity $1/\omega$ (ω is a frequency of oscillations) is used. A key idea of Bolotin method can be briefly described in the following way. For boundary conditions of simply supporting the LNM for beams, plates and shallow shells can be easily constructed. For other boundary conditions these LNM create the asymptotic of searching solutions in an interior zone. In the vicinity of the boundaries dynamical edge effects take place. In interior zone of continuous system solutions can be expressed with trigonometric functions with unknown constants and one can use exponential functions in the dynamic edge effect's zone. Then matching procedure permits to obtain unknown constants and LNM can be expressed in relatively simple forms. This approximate solution is very accurate for high frequency oscillations, but even for low frequency oscillation the error is not excessive. According to the boundary conditions:

$$W_0=A\sin\frac{\pi(s-s_0)}{n}x(t) \qquad (4)$$

where A is amplitude, s_0 is phase shift and n is wave number. Equation for time function x(t) is as follows:

$$\frac{d^2x(t)}{dt^2}+\omega^2(1+\gamma x^2(t))x(t)=0 \qquad (5)$$

Where

$$\omega^2=\frac{\pi^2 E r^2}{\rho\lambda^2},$$

$$\gamma=\frac{0.25(1+k)A^2}{r^2},$$

$$k=\frac{n}{2\pi L}[\sin\frac{2\pi(1-x_0)}{n}+\sin\frac{2\pi x_0}{n}],$$

and r^2 = I/F. With some manipulation in above equation:

$$k=\frac{n}{2\pi L}[\sin\frac{2\pi(1-x_0)}{n}+\sin\frac{2\pi x_0}{n}], \qquad (6)$$

Where $\beta=\alpha\gamma$, $\alpha=\omega^2$. In the linear case ($\gamma=0$) depends upon boundary equation (3) has solutions as sine or cosine functions (normal forms of linear oscillations). For example, for boundary conditions:

$$x(0)=\lambda,\ \ x'(0)=0 \qquad (7)$$

Solution has the form $\cos(\sqrt{\alpha}\,t)$. As we deal with nonlinear generalization of normal forms it is natural to use as example the same boundary conditions (6). Dynamical edge effect component may be introduced as follows:

$$W=W_0+W_e \qquad (8)$$

We can suppose that length of beam L is enough for possibility to omit mutual influences of edge effects near the boundaries x = 0 and x = L. Then the boundary conditions for edge effect near the boundary x = 0 can be replaced by the condition of decaying:

$$W_e\to 0 \qquad (\text{for } x\to\infty) \qquad (9)$$

APPLICATION OF HAM

The governing equation for the nonlinear beam is expressed by Eq. (6) and Eq. (7). Nonlinear operator is defined as follow:

$$N[x(t;q)] = \frac{\partial^2 x(t;q)}{\partial t^2} + \alpha x(t;q) + \beta x(t;q)^3 \tag{10}$$

where $q \in [0,1]$ is the embedding parameter. As the embedding parameter increases from *0* to *1*, $U(t;q)$ varies from the initial guess, $U_0(t)$, to the exact solution, $U(t)$.

$$x(t;0) = U_0(t), \qquad x(t;1) = U(t) \tag{11}$$

Expanding x(t;q) in Taylor series with respect to q results in:

$$x(t;q) = U_0(t) + \sum_{m=1}^{\infty} U_m(t) q^m \tag{12}$$

Where:

$$U_m(t) = \frac{1}{m!} \frac{\partial^m x(t;q)}{\partial q^m}\bigg|_{q=0} \tag{13}$$

Homotopy analysis method can be expressed by many different base functions [4], according to the governing equations; it is straightforward to use a base function in the form of:

$$U(t) = \sum_{m=1}^{\infty}\sum_{p=1}^{\infty} b_{kpm} t^k \cos^m(t) \sin^p(t) \tag{14}$$

That b_{kpm} are the coefficients to be determined. When the base function is selected, the auxiliary functions $H(t)$, initial approximations $U_0(t)$ and the auxiliary linear operators L must be chosen in such a way that the corresponding high-order deformation equations have solutions with the functional form similar to the base functions. This method referred to as the rule of solution expression [4]. The linear operator L is chosen as:

$$L[x(t;q)] = \frac{\partial^2 x(t;q)}{\partial t^2} + x(t;q) \tag{15}$$

From Eq. (14) results in:

$$L[c_1 \sin(t) + c_2 \cos(t)] = 0 \tag{16}$$

where c_1 and c_2 are the integral constants. According to the rule of solution expression and the initial conditions, the initial approximations, U_0 as well as the integral constants, c_1 to c_2 are formed as:

$$U_0(t) = c_1 \sin(t) + c_2 \cos(t), \quad c_1 = 0, c_2 = \lambda \tag{17}$$

The zeroth order deformation equation for $U(t)$ is:

$$(1-q)L[x(t;q) - U_0(t)] = q\hbar H(t) N[x(t;q)] \tag{18}$$

$$x(0;q) = \lambda, \qquad \frac{\partial x(0;q)}{\partial t} = 0 \tag{19}$$

According to the rule of solution expression and from Eq. (18), the auxiliary function $H(t)$ can be chosen as follows:

$$H(t) = 1 \tag{20}$$

Differentiating Eq. (18), m times, with respect to the embedding parameter q and then setting $q=0$ in the final expression and dividing it by $m!$, it is reduced to:

$$U_m(t) = \hbar \begin{pmatrix} \sin(t)\int_0^t H(t) R_m(U_{m-1})\cos(t)dt \\ +\cos(t)\int_0^t H(t) R_m(U_{m-1})\sin(t)dt \end{pmatrix} + c_1 \sin(t) + c_2 \cos(t) + \chi_m U_{m-1}(t) \tag{21}$$

$$U_m(0) = 0, \qquad U'_m(0) = 0 \tag{22}$$

Eq. 17 is the mth order deformation equation for x(t), where

$$R_m(U_{m-1}) = \frac{d^2 U_{m-1}(t)}{dt^2} + \alpha U_{m-1}(t) + \beta\left[\sum_{j=0}^{m-1} U_{m-1-j}(t) \sum_{z=0}^{j} U_{j-z}(t) U_z(t)\right] \tag{23}$$

And

$$\chi_m = \begin{cases} 0, & m \le 1 \\ 1, & m > 1 \end{cases} \tag{24}$$

As a result of this selection, the first and second terms of the solution's series are as follows:

$$U_0(t) = \lambda \cos(t) \tag{25}$$

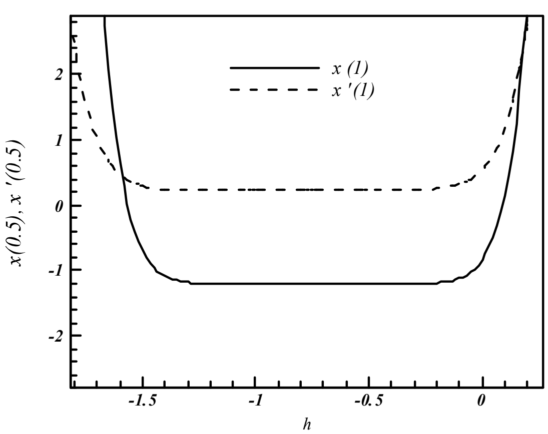
$$U_1(t) = \frac{1}{8}\sin(t) h \lambda(-4t + 4\alpha t + 3\lambda^2 \beta t + \sin(t)\cos(t)\lambda^2\beta) \tag{26}$$

The solution's series $U(t)$ is developed up to 12th order of approximation.

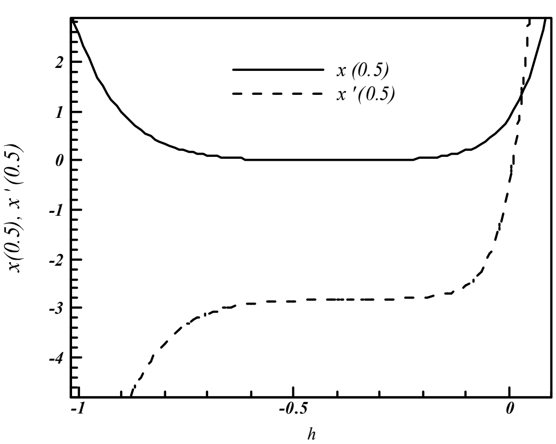
CONVERGENCE OF HAM SOLUTION

The analytical solution should converge. It should be noted that the auxiliary parameter h, as pointed out by Liao [3, 4], controls the convergence and accuracy of the solution series. In order to define a region such that the solution series is independent on h, a multiple of h -curves are plotted. The region where the distribution of $x'(t)$ and x (t) versus h is a horizontal line is known as the convergence region for the corresponding function. The common region among the x (t) and its derivatives are known as the overall convergence region.

To study the influence of h on the convergence of solution, the h -curves of $x'(1)$, x (1) and $x'(0.5)$, x (0.5) are plotted for selected values of constant numbers, as shown in Fig.1. Moreover, increasing the order of approximation increases the range of the convergence region, as shown in Fig.2.

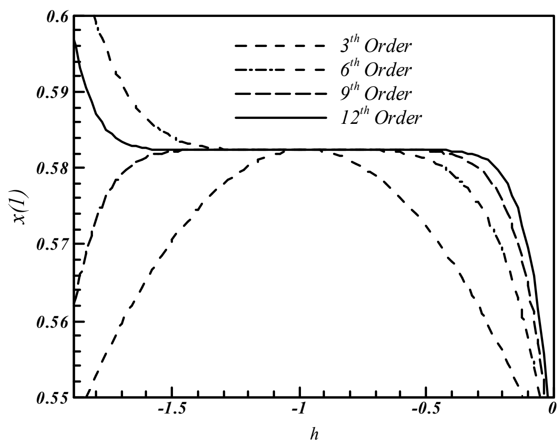


(a)

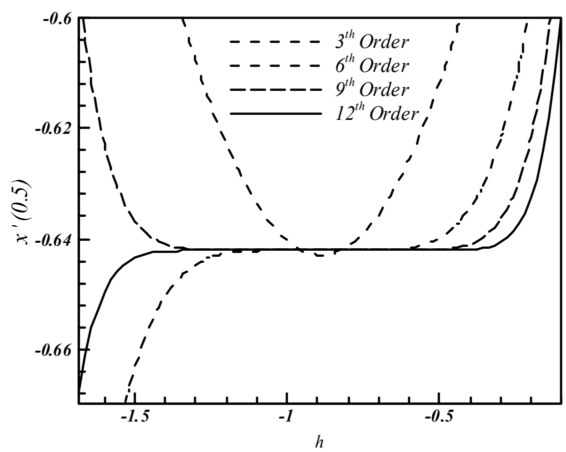


(b)

Fig. 1: The *h* curves to indicate the convergence region, (a) $\alpha = 1$, $\beta = 1$, (b) $\alpha = 2$, $\beta = 2$



(a)



(b)

Fig. 2: The effect of order of approximation on convergence region, (a) $\alpha = 0.5$, $\beta = 3$, (b) $\alpha = 0.5$, $\beta = 5$

RESULTS AND DISCUSSION

Eq. (6) is solved using HAM for some values of the parameters *a* and *ß*. The results for *x(t), x'(t)* and W_0 at some values of *t* are shown in Table 1-5. The results obtained from HAM solution of *x(t)* and *x'(t)* are compared with results of numerical solution based on fourth order Runge Kutta method. It is worth mentioning that the relative error is defined as follows:

$$E_{rel} = 100 \times \left| 1 - \frac{\text{results of HAM}}{\text{Numericalresults}} \right| \quad (27)$$

The effects of constant parameters, α and β, on the response of system, velocity function depend on *t* have been shown in Fig. 3.

The nonlinear differential equation resulting from nonlinear normal modes for continuous systems is

Table 1: Compression between results of x (t) predicted by HAM and numerical method: $\gamma = 1$, $\omega = 0.5$

t	HAM	Numerical	Relative error
0.1	0.997033	0.997502	0.046938
0.3	0.977689	0.977667	0.002294
0.5	0.938799	0.938767	0.003419
0.8	0.848676	0.847980	0.082036
1.0	0.770000	0.768801	0.155848
1.5	0.523677	0.522419	0.240861
2.0	0.234463	0.233691	0.330251
3.0	-0.374445	-0.371030	0.920450
4.0	-0.849999	-0.859349	1.087954
5.0	-0.998334	-0.986606	1.188714

Table 2: Compression between results of x′ (t) predicted by HAM and numerical method: $\gamma = 1$, $\omega = 0.5$

t	HAM	Numerical	Relative error
0.1	-0.049926	-0.049916	0.019508
0.3	-0.147837	-0.147782	0.037341
0.5	-0.239876	-0.239991	0.047702
0.8	-0.361599	-0.361382	0.059456
1.0	-0.428765	-0.428447	0.074147
1.5	-0.545876	-0.545397	0.087858
2.0	-0.601801	-0.600811	0.164780
3.0	-0.582987	-0.581562	0.245019
4.0	-0.351006	-0.349585	0.406059
5.0	0.115897	0.114957	0.817652

Table 3: Compression between results of x (t) predicted by HAM and numerical method: $\gamma = 2$, $\omega = \pi$

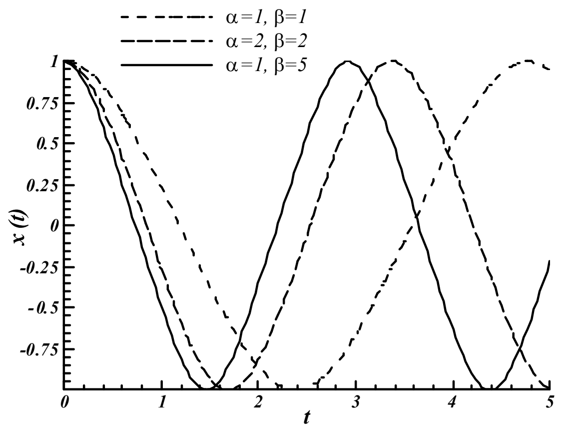
t	HAM	Numerical	Relative error
0.1	0.859800	0.859893	0.010821
0.3	0.082813	0.082824	0.014074
0.5	-0.747324	-0.747485	0.021489
0.8	-0.658321	-0.658082	0.036241
1.0	0.195112	0.195033	0.040644
1.5	0.407600	0.407862	0.064336
2.0	-0.889209	-0.889990	0.087783
3.0	-0.567988	-0.566981	0.177690
4.0	0.609942	0.607336	0.429004
5.0	0.869981	0.863981	0.694416

Table 4: Compression between results of x′ (t) predicted by HAM and numerical method: $\gamma = 2$, $\omega = \pi$

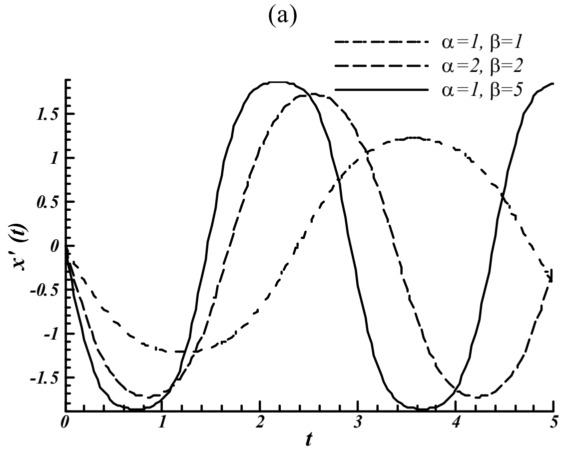
t	HAM	Numerical	Relative error
0.1	-2.654899	-2.654315	0.022002
0.3	-4.436309	-4.435203	0.024939
0.5	-3.336230	-3.338196	0.058919
0.8	3.687112	3.686971	0.003821
1.0	4.394889	4.398804	0.089000
1.5	-4.229970	-4.221874	0.191773
2.0	2.382007	2.393623	0.485284
3.0	-3.968994	-3.942889	0.662068
4.0	-3.872254	-3.841329	0.805203
5.0	2.648891	2.621500	1.044832

Table 5: Compression between results of W_0 predicted by HAM and numerical method: A = 0.02, n = 1, $s_0 = 0$

s	T	HAM	Numerical	Relative error
0.1	0.1	0.006161	0.006165	0.061273
0.1	0.3	0.006047	0.006042	0.075322
0.3	0.5	0.015174	0.015190	0.102581
0.3	0.8	0.013688	0.013721	0.240262
0.5	1.0	0.015424	0.015376	0.310480
0.5	1.5	0.010500	0.010448	0.495124
0.75	2.0	0.033299	0.033049	0.756346
0.75	3.0	-0.005300	-0.005250	0.983767
0.9	4.0	-0.005380	-0.005310	1.254004
0.9	5.0	-0.006010	-0.006100	1.602937



(a)



(b)

Fig. 3: The effects of small parameter on the response, 10 orders of approximation, (a) x (t), (b) x′ (t)

studied using Homotopy Analysis Method. The comparison with numerical results and convergence study shows that using approximations of small orders results in satisfactory accuracy and increasing the number of order the accuracy is increased.

The proposed analytical approach has many applications and thus may be applied in similar ways to other oscillations to get accurate series solutions.

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