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Approximate Solution Corresponding to Periodically Heterogeneous Love-Kirchhoff Plate

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Abstract: In this paper, we give an approximate mathematical model of a periodically non-homogeneous Love-Kirchhoff plate with periodic inclusions or perforations. Dealing with the variational formulation, we prove the existence and the uniqueness of a weak solution. The asymptotic expansion is used to derive the homogenized approximate problem and the first correction term. The use of the finite element method in such case is very complicated. The homogenization method [1-6] leads to replace this complicated finite element problem in the whole domain Ω by two simple problems in one cell Y.

Key words: Boundary value problem · Approximate solutions · Homogenization theory

INTRODUCTION

In the study of some composite materials, in Mec, Phy and Engineering one is led to the study of boundary value problem in media with periodic structure. If the period of the structure is very small compared to the size of the region in witch the system is to be studied (i.e. if the number of heterogeneities is very large). For solving such problem numerically by using fore example F.E.M the periodic medium must be divided into a very hunk small elements. Moreover, this number increases infinitely when the number of the cells becomes very big, that is when the length of the period becomes very small. Therefore the use of the F.E.M in this case is very complicated. The homogenization method consists in replacing the heterogeneous medium (microscopic) by an "equivalent" homogeneous (macroscopic) one. The homogenization method has been employed with full advantage in solving boundary value problem in non-homogeneous media with periodic structures [1-2]. The homogenization method of a plate with periodic structures was treated in detail [1-8], but only in the case of thin plate, However, the correction terms are not available.

Differential Form Governing: the displacements in periodically non-homogeneous Love-Kirchhoff plate. According to the Love-Kirchhoff hypotheses (the linear theory of thin sheets that are not subject to shear stresses) The mathematical model of non-homogenous isotropic Love-Kirchhoff plate with periodic inclusions (consist of two parts i.e. major and minor) can be represented by the following boundary value:

$$\begin{cases} \frac{\partial 2}{\partial x_i\, \partial x_j}\left({}^{\varepsilon} a^{ij}_{kl} \frac{\partial 2}{\partial x_k \partial x_l}\right) u^{\varepsilon} = f_D & in \quad \Omega \\ u^{\varepsilon} = \frac{\partial u^{\varepsilon}}{\partial n} = 0 & on \partial\Omega_D \\ F(u^{\varepsilon}) = C(u^{\varepsilon}) = 0 & on \partial\Omega_N \\ \partial\Omega = \partial\Omega_L \cup \partial\Omega_N & f_D = \frac{f}{D} \qquad D = \frac{Ee^3}{12(1-\nu)} \end{cases} \tag{1}$$

where D is a rigidity coefficient, ν Poisson,s ratio and e the thickness of the plate.

The periodicity of the plate can be expressed by the periodicity of the coefficients which can be expressed by the speed variable $y = \frac{x}{\varepsilon}$ in the forms: ${}^{\varepsilon} a^{ij}_{kl}(x) = a^{ij}_{kl}(y)$

where x is macroscopic quantity and $y = \frac{x}{\varepsilon}$ is a microscopic one, we define the cell by $Y = \prod_{i=1}^{2} \left[0, Y_i\right]$

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Variational Formulation: The weak formulation of the problem (1) is given by:

$$\begin{cases} Find \quad u^{\varepsilon} \in V \quad such \quad that \\ a(u^{\varepsilon},v)_V \;= (f_D,v)_{L^2(\Omega)} \forall v \in V \end{cases} \quad (2)$$

where:

$$a(u^{\varepsilon},v)_V \;= \int_{\Omega} \varepsilon\, a_{kl}^{ij} \frac{\partial^2 u}{\partial x_k \partial x_l}\frac{\partial^2 v}{\partial x_i \partial x_j} dx$$

$$V \;= \left\{ v \in H^2(\Omega); tracev \Big|_{\partial\Omega_D} = 0 \quad trace \frac{\partial v}{\partial n} \;\Big|_{\partial\Omega_D} = 0 \right\}$$

and $H^2(\Omega) = \left\{ v; D^{\alpha} v \in L^2(\Omega) \forall \alpha; |\alpha| \le 2 \right\}$

It is clear that the space V is a Hilbert space with the scalar product:

$$(u,v)_V \;= \int_{\Omega} \partial^2 u \partial^2 v = \int_{\Omega} \frac{\partial^2 u}{\partial x_k \partial x_l}\frac{\partial^2 v}{\partial x_i \partial x_j} dx$$

(a closed subspace of $H^2(\Omega)$).

By Applying Green formula and using the boundary condition, one prove the equivalent between the differential form (1) and the variational form (2).

The bilinear form verified the inequality:

$$\alpha |u|^2{}_V < a(u,u) < \beta |u|^2{}_V$$

This means that this bilinear form is positive definite on V(V-coercive or V-elliptic and bounded on V), therefore, t he Lax-Milgram lemma prove that the solution of (2) exist and it is unique.

Asymptotic Expansions: Using the technique of homogenization method [1-3], the solution of the weak form (2) can be expressed as an asymptotic expansions:

$$u^{\varepsilon}(x) = u_0 \;(x) + \varepsilon u_1 \;(x,y) + \varepsilon^2 u_2 \;(x,y) + \quad (3)$$

where $u_i(x,y)$ is Y-periodic in y.

Substituting (3) in (2) and taking into account the composite derivations:

$$\frac{d}{dx_i} = \frac{\partial}{\partial x_i} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_i},$$

$$\frac{d^2}{dx_i dx_j} = \frac{\partial^2}{\partial x_i \partial x_j} + \frac{1}{\varepsilon}\left(\frac{\partial^2}{\partial x_i \partial x_j} + \frac{\partial^2}{\partial x_i \partial x_j}\right) + \frac{1}{\varepsilon^2}\frac{\partial^2}{\partial x_i \partial x_j}$$

then by grouping the terms of the same orders in *ε*, we can write (2) in the form:

$$\begin{cases} \varepsilon^{-4} B_{-4}(x,y) \;+ \varepsilon^{-3} B_{-3}(x,y) + \varepsilon^{-2} B_{-2}(x,y) \;+ \\ \varepsilon^{-1} B_{-1}(x,y) \;+ \varepsilon^{0} B_0(x,y) + \varepsilon B_1(x,y) .....+ = \int_{\Omega} f_D.vdx \end{cases} \quad (4)$$

Passing to the limit when *ε* tends to zero, applying the mean lemma and if we compare the terms of the same orders in *ε* we obtain the following equations whith respect to the functions $u_i(x,y)$, I=0,1,2,……n

$$\lim_{\varepsilon \to 0} B_0(x,y) = \int_{\Omega} f_D.vdx$$

$$\lim_{\varepsilon \to 0} B_i(x,y) = o \quad i = -1,-2,-3,-4$$

The Problem on $u_0(x,y)$: This problem can be obtained by taking the limit

$$\lim_{\varepsilon \to 0} \int_{\Omega} a_{ijkl}(y) \frac{\partial^2 u_0}{\partial y_k \partial y_l} . \frac{\partial^2 v}{\partial y_i \partial y_j} dy = 0 \quad (5)$$

Since the test function v must be periodic with respect to the variable y, it can be chosen from the form:

$$v = v(x).w(y), \; v(x) \in D_D(\Omega), w(y) \in H_Y^2(Y)$$

where:

$$D_D(\Omega) = \left\{ v \in D_D(\overline{\Omega}); v \quad and\, all \quad their \quad derivatives = 0 \quad on\, \partial\Omega_D \right\}$$

$$H_Y^2(Y) = \left\{ \begin{array}{l} w \in H^2(Y); are Y-periodic \quad taking \quad the \\ same\, value\, on\, the\, opposite\, faces\, of\, Y \cdots \\ \\ and \quad \tilde{w} = \frac{1}{Y}\int_Y w dy = 0; \tilde{w} = average(mean) value \end{array} \right\}$$

Then we have the following problem on $u_0(x,y)$:

$$\begin{cases} Find \quad u_0 \in H_Y^2(Y) \quad such \quad that \\ \int_Y a_{ijkl}(y)\frac{\partial 2u_0}{\partial y_k \partial y_l}.\frac{\partial 2w}{\partial y_i \partial y_j}dy = 0 \forall w \in H_Y^2(Y) \end{cases} \quad (6)$$

The preceding problem (6) accepts a periodic solution independent of y, that is, the function u_0 is constant with respect to y That is $u_0 = u(x)$ Similarly, we get a similar result for $u_1 = u_1(x)$

The Problem on $u_2(x,y)$: This problem can be obtained by solving the equation

$$\lim_{\varepsilon \to 0} B_{-2}(\varepsilon) = 0$$

witch can be written in the following weak form:

$$\begin{cases} Find \quad u_2 \in H_Y^2(Y) \, such \quad that \\ \int_Y a_{ijkl}(y)\frac{\partial 2(u_2+u_0)}{\partial y_k \partial y_l}.\frac{\partial 2w}{\partial y_i \partial y_j}dy = 0, \forall w \in H_Y^2(Y) \end{cases} \quad (7)$$

Then by virtue the linearity of the previous boundary values problem, it is possible to search for the function $u_2(x,y)$ as the from

$$u_2(x,y) = W^{mn}(y)\frac{d^2u_0}{dx_m dx_n} + c(x)$$

where w^{mn} Y-periodic taking the same value on the opposite faces of the boundary of the cell Y, this condition plays the role of boundary condition verified the following local boundary value problem.

$$\begin{cases} Find \quad W^{mn} \in H_Y{}^2(Y) \quad such \quad that \\ \int_Y a_{ijkl}(y)\frac{d^2W^{kl}}{dy_k dy_l}.\frac{d^2v}{dy_i dy_j}dy = \\ -\int_Y a_{ijmn}(y).\frac{d^2v}{dy_i dy_j}dy \\ \forall v \in H_Y{}^2(Y) \end{cases} \quad (8)$$

The Homogenized Problem: The boundary problem representing the mean behavior (homogeneous problem) is written in the following variational (weak) form:

$$\begin{cases} Find \quad u_0 \in V \, such \quad that \\ \int_\Omega a^h{}_{ijkl}(y)\frac{d^2u_0}{dx_k dx_l}.\frac{d^2v}{dx_i dx_j} = \int_\Omega f_D v dx \quad \forall v \in V \end{cases} \quad (9)$$

where:

$$a^h{}_{ijkl} = \frac{1}{|Y|}\int_Y [a_{ijkl}(y) - a_{ijmn}(y)\frac{d^2W^{kl}(y)}{dy_m dy_n}]dy \quad (10)$$

The relationship between the solution of the non-homogenous problem (local problem)and the solution of the homogeneous problem (global problem) can be established by following proposition:

Proposition: The solution of initial problem u^ε converge weakly to as u^0 tends to zero.

Proof: By using the Poincare inequality, V-elliptic Characteristic and Schwartz inequality respectively we have the following inequalities:

$$\begin{cases} c\left\|u^\varepsilon\right\|_V^2 \le c_1 \int_\Omega (\frac{\partial^2 u^\varepsilon}{\partial x_i \partial x_j})^2 dx \le a(u^\varepsilon, u^\varepsilon) = \\ (f_D, v) \le c_2 \left\|u^\varepsilon\right\|_{L^2(\Omega)} \le c_3 \left\|u^\varepsilon\right\|_V \end{cases} \quad (10)$$

As a result we get:

$$\left\|u^\varepsilon\right\|_V \le K > 0$$

Therefore

$$u^\varepsilon \to U \quad weakly \quad in \quad V$$

Whereas, the homogeneous problem verified the conditions of Lax-Milgram lemma, then it accepts a unique solution and from it follows that:

$$U = u^0$$

CONCLUSION

It follows from the previous results that the homogenization method is an effective method for solving boundary value problem in a heterogeneous medium (periodically non-homogeneous plate). It leads to replace a very (complicated BVP "numerically expensive") in the whole domain corresponding to periodically inclusion (specifically when the number of the cells is very big that is when the length of the period becomes very small) by two simple BVP the first defined in a representative cell and the second in whole domain with constant coefficients who can be solved easily. Furthermore the case of the perforated plate can be considered as limit case.

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