

Decomposition Method for Fractional Partial Differential Equations Using Modified Integral Transform

¹Tarig M. Elzaki, ²Yassir Daoud and ³J. Biazar

¹Mathematics Department, Faculty of Sciences and Arts,
 Alkamil University of Jeddah, Jeddah-Saudi Arabia

²Mathematics Department, Faculty Mathematical Sciences and Statistics,
 Alneelain University, Khartoum, Sudan

³Department of Applied Mathematics,
 Faculty of Mathematical Sciences, University of Guilan, Rasht, Iran

Abstract: In this research, an effective combination of the result of fractional partial differential equations (PDEs) is envisioned. Decomposition coupled with modified integral transform (Elzaki transform) is applied to solve partial differential equations, of a fractional order. It is observed that the proposed technique is extremely useful. The effects of the proposed scheme are highly encouraging and efficient.

Key words: Elzaki transform method • Fractional differential equations • Wave equation • Burgers equation • Fluid mechanics.

INTRODUCTION

During the last years, many researchers found that the derivatives of non-integer order are very suited for the description of various physical phenomena such as dumping laws, diffusion process, etc. These findings raised the growing interest of studies of fractional calculus in several areas such as natural philosophy, alchemy and technology. For these causes, we require honest and effective techniques for the solution of fractional differential equations [1-9, 11, 13, 14]. Most fractional differential equations do not take an exact analytic solution, therefore approximation and numerical techniques must be employed. In the last decades, various methods have been employed to solve fractional differential equations, fractional partial differential equations, fractional integro-differential equations [2, 6, 10, 12, 15] and dynamic systems containing fractional derivatives, such as ottomans decomposition method [2, 610, 12, 15], He's a variational iteration method [12], Homotopy perturbation method [9, 16]. Laplace transform method [1, 4, 5]. In this article, we use the decomposition method coupled with Elzakitrans form [17-20] to construct

appropriate solutions to multi-dimensional wave, Burger's and Klein-Gordon equations of fractional order.

Definition and Derivations the ELzaki Transform of

Derivatives: ELzaki transform of the function $f(t)$ is defined as;

$$E[f(t)] = T(v) = v \int_0^{\infty} f(t) e^{-t/v} dt, \quad t > 0, \quad v \in (k_1, k_2)$$

To obtain the ELzaki transform of partial derivatives we use integration by parts as follows:

$$E\left[\frac{\partial f(x,t)}{\partial t}\right] = \int_0^{\infty} v \frac{\partial f}{\partial t} e^{-t/v} dt = \lim_{p \rightarrow \infty} \int_0^p v e^{-t/v} \frac{\partial f}{\partial t} dt =$$

$$\lim_{p \rightarrow \infty} \left\{ \left[v e^{-t/v} f(x,t) \right]_0^p - \int_0^p e^{-t/v} f(x,t) dt \right\} = \frac{T(x,v)}{v} - v f(x,0)$$

we assume that f is piecewise continuous and is of exponential order.

Right away

$$E\left[\frac{\partial f}{\partial x}\right] = \int_0^\infty v e^{-\frac{t}{v}} \frac{\partial f(x,t)}{\partial x} dt = \frac{\partial}{\partial x} \int_0^\infty v e^{-\frac{t}{v}} f(x,t) dt = \frac{\partial}{\partial x} [T(x,v)]$$

Then we get:

$$E\left[\frac{\partial f}{\partial x}\right] = \frac{d}{dx} [T(x,v)]$$

Similarly, we can recoup:

$$E\left[\frac{\partial^2 f}{\partial x^2}\right] = \frac{d^2}{dx^2} [T(x,v)]$$

Let us take $\frac{\partial f}{\partial t} = g$, then we have:

$$E\left[\frac{\partial^2 f(x,t)}{\partial t^2}\right] = E\left[\frac{\partial g(x,t)}{\partial t}\right] = \frac{1}{v} E[g(x,t)] - v g(x,0)$$

$$E\left[\frac{\partial^2 f(x,t)}{\partial t^2}\right] = \frac{1}{v^2} T(x,v) - f(x,0) - v \frac{\partial f}{\partial t}(x,0)$$

We can well extend this result to the n^{th} partial derivative by using mathematical induction,

See [14, 15, 16] for more details and examples.

Analysis of the Proposed Scheme: To illustrate the basic idea of the proposed method we study a general nonlinear non homogeneous fractional partial differential equation with initial conditions at the origin:

$$D_t^\alpha u(x,t) + Ru(x,t) + Nu(x,t) = g(x,t), \quad 0 < \alpha \leq 2 \quad (1)$$

$$u(x,0) = h(x), \quad u_t(x,0) = f(x), \quad (2)$$

where $g(x,t)$ is the source term, N represents an on linear operator and R is a linear differential operator, where $h(x), f(x)$ are algebraic functions and $D_t^\alpha u(x,t)$ is the Caputo fractional derivative of the function $u(x,t)$ which is definite by,

$$D_t^\alpha u(x,t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{u^{(n)}(x,\zeta)}{(t-\zeta)^{\alpha+1-n}} d\zeta, \quad n-1 < \alpha \leq n, \quad (3)$$

Elzakitrans form of the Caputo operatoris,

$$E[D_t^\alpha u(x,t)] = \frac{1}{v^\alpha} E[u(x,t)] - \sum_{k=0}^{n-1} u^{(k)}(x,0) v^{2-\alpha+k} \quad (4)$$

Taking Elzaki transform of Equation (1) results in:

$$E[D_t^\alpha u(x,t)] + E[Ru(x,t)] + E[Nu(x,t)] = E[g(x,t)], \quad (5)$$

Use the property of Elzaki transform, toget:

$$E[u(x,t)] = v^\alpha h(x) + v^{\alpha+1} f(x) - v^\alpha E[Ru(x,t)] - v^\alpha E[Nu(x,t)] + v^\alpha E[g(x,t)], \quad (6)$$

Operating with Elzaki inverse on both sides of Equation (6) gives:

$$u(x,t) = G(x,t) - E^{-1} \{ v^\alpha E[Ru(x,t)] + v^\alpha E[Nu(x,t)] \}, \quad (7)$$

Where $G(x,t)$ represents the term arising from the source term and the prescribed in initial conditions. Using Adomians Decomposition Method (ADM) to get out the nonlinear terms, then the solution will be in the following form:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t). \quad (8)$$

Numerical Applications: We gave the suggested method to engender to the effect of the fractional partial differential equation (PDEs). Numerical results obtained from the proposed system are advanced. To match the efficiency, few examples are brought in:

Example 1: Look at the following one-dimensional linear fractional wave equation:

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \frac{\partial u}{\partial x} = \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \sin x + t \cos x, \quad t > 0, \quad 0 < \alpha \leq 1, \quad (9)$$

Subject to the initial condition,

$$u(x,0) = 0$$

Applying Elzaki transform onequation (9) results in,

$$E\left[\frac{\partial^\alpha u}{\partial t^\alpha} + \frac{\partial u}{\partial x}\right] = E\left[\frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \sin x + t \cos x\right],$$

$$\frac{1}{v^\alpha} E[u(x, t)] - v^{2-\alpha} u(x, 0) = E\left[\frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \sin x + t \cos x\right] - E\left[\frac{\partial u}{\partial x}\right],$$

$$E[u(x, t)] = v^\alpha E\left[\frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \sin x + t \cos x\right] - v^\alpha E\left[\frac{\partial u}{\partial x}\right],$$

Using the Adomian decomposition method, we have,

$$E[u_0(x, t)] = v^\alpha E\left[\frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \sin x + t \cos x\right],$$

$$E[u_{k+1}(x, t)] = -v^\alpha E\left[\frac{\partial u_k}{\partial x}\right], \quad k \geq$$

Then we get,

$$E[u_0(x, t)] = v^\alpha \frac{\Gamma(2-\alpha)}{\Gamma(2-\alpha)} v^{3-\alpha} \sin x + v^{3+\alpha} \cos x,$$

$$E[u_0(x, t)] = v^3 \sin x + v^{3+\alpha} \cos x \quad \Rightarrow \quad u_0(x, t) = t \sin x + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \cos x$$

$$E[u_1(x, t)] = -v^{3+\alpha} \cos x + v^{3+2\alpha} \sin x \quad \Rightarrow \quad u_1(x, t) = -\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \cos x + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} \sin x$$

$$u_2(x, t) = -\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} \sin x - \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)} \cos x, \dots$$

And so the solution in series configuration is presented in the form,

$$u(x, t) = t \sin x + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \cos x - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \cos x + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} \sin x$$

$$- \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} \sin x - \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)} \cos x + \dots$$

Striking down the noise terms and holding open the non-noise terms yield the accurate result of equation (9), in the form,

$$u(x, t) = t \sin x$$

Example 2: Consider the following one-dimensional linear fractional Burger's equation:

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} = \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} + 2x - 2, \quad x \in R, \quad 0 < \alpha \leq 1, \quad (10)$$

subject to initial condition,

$$u(x, 0) = x^2$$

Applying Elzaki transform on both sides of equation (10) results in,

$$E\left[\frac{\partial^\alpha u}{\partial t^\alpha} + \frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2}\right] = E\left[\frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} + 2x - 2\right],$$

$$\frac{1}{v^\alpha} E[u(x, t)] - v^{2-\alpha} u(x, 0) = E\left[\frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} + 2x - 2\right] - E\left[\frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2}\right]$$

$$E[u(x, t)] = v^2 x^2 + v^\alpha E\left[\frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} + 2x - 2\right] - v^\alpha E\left[\frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2}\right].$$

By using the Adomian decomposition method, we receive the following recurrence relation,

$$E[u_0(x, t)] = v^2 x^2 + v^\alpha E\left[\frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} + 2x - 2\right]$$

$$E[u_{K+1}(x, t)] = -v^\alpha E\left[\frac{\partial u_k}{\partial x} - \frac{\partial^2 u_k}{\partial x^2}\right], \quad k \geq 0$$

And then we give up,

$$E[u_0(x, t)] = v^2 x^2 + v^\alpha \left[\frac{2\Gamma(3-\alpha)v^{4-\alpha}}{\Gamma(3-\alpha)} + 2v^2 x - 2v^2 \right] \Rightarrow u_0(x, t) = t^2 + x^2 + \frac{2xt^\alpha}{\Gamma(\alpha+1)} - \frac{2t^\alpha}{\Gamma(\alpha+1)},$$

$$E[u_1(x, t)] = 2v^{\alpha+2} - 2xv^{\alpha+2} - 2v^{2\alpha+2} \Rightarrow u_1(x, t) = \frac{2t^\alpha}{\Gamma(\alpha+1)} - \frac{2xt^\alpha}{\Gamma(\alpha+1)} - \frac{2t^{2\alpha}}{\Gamma(2\alpha+1)},$$

And then the series solution will be given as follows,

$$u(x, t) = t^2 + x^2 + \frac{2xt^\alpha}{\Gamma(\alpha+1)} - \frac{2t^\alpha}{\Gamma(\alpha+1)} + \frac{2t^\alpha}{\Gamma(\alpha+1)} - \frac{2xt^\alpha}{\Gamma(\alpha+1)} - \frac{2t^{2\alpha}}{\Gamma(2\alpha+1)} + \dots$$

$$u(x, t) = t^2 + x^2 - \frac{2t^{2\alpha}}{\Gamma(2\alpha+1)} + \dots$$

Coming down the noise terms, to see the exact solution in the physical body, $u(x, t) = t^2 + x^2$

Example 3: Let's look at the following one-dimensional linear fractional Klein-Golden equation:

$$\frac{\partial^\alpha u}{\partial t^\alpha} - \frac{\partial^2 u}{\partial x^2} + u = 6x^3 t + (x^3 - 6x)t^3, \quad t > 0, \quad 0 < \alpha \leq 2, \quad (11)$$

subject to the initial conditions,

$$u(x, 0) = 0, \quad u_t(x, 0) = 0.$$

Applying Elzaki transform on both sides of equation (11), to get,

$$E\left[\frac{\partial^\alpha u}{\partial t^\alpha} - \frac{\partial^2 u}{\partial x^2} + u\right] = E[6x^3 t + (x^3 - 6x)t^3],$$

$$E[u(x, t)] = v^\alpha E[6x^3 t + (x^3 - 6x)t^3] + v^\alpha E\left[\frac{\partial^2 u}{\partial x^2} - u\right].$$

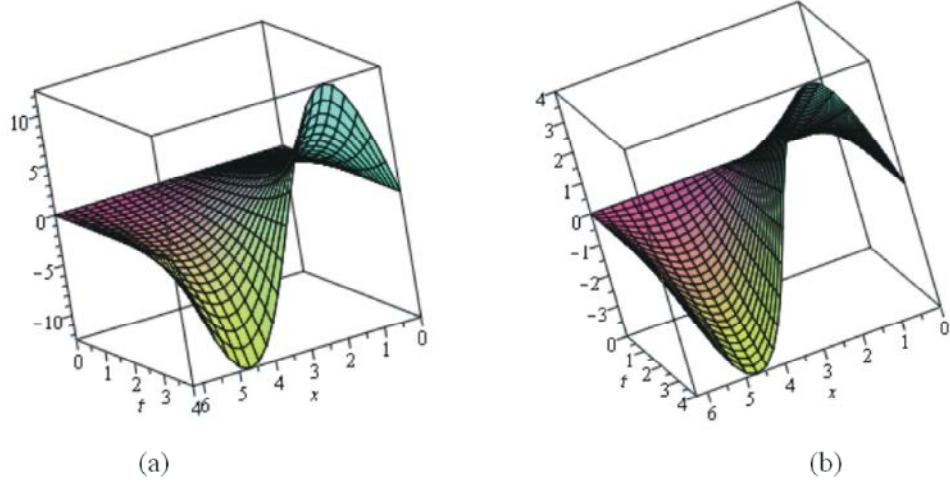


Fig. 1: (a), Approximate solution, (b), Exact solution

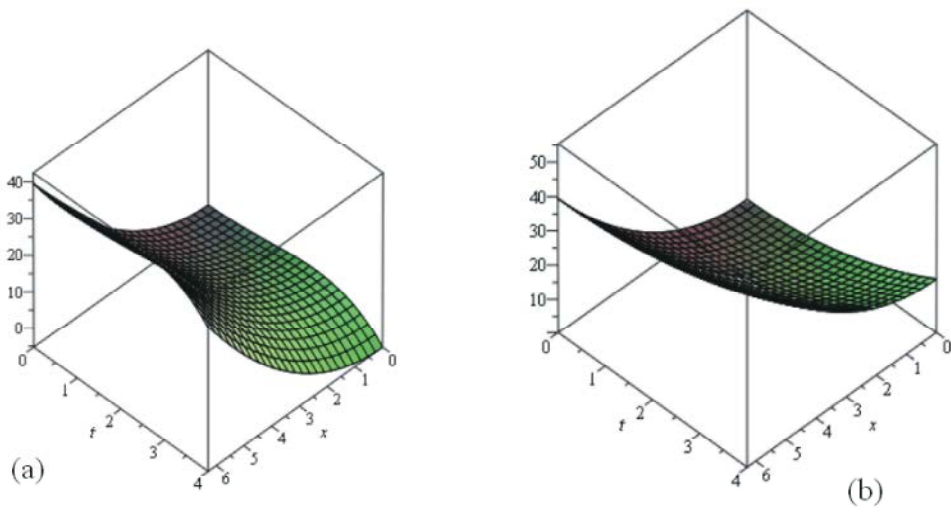


Fig. 2: (a):Approximate solution, (b):Exact solution

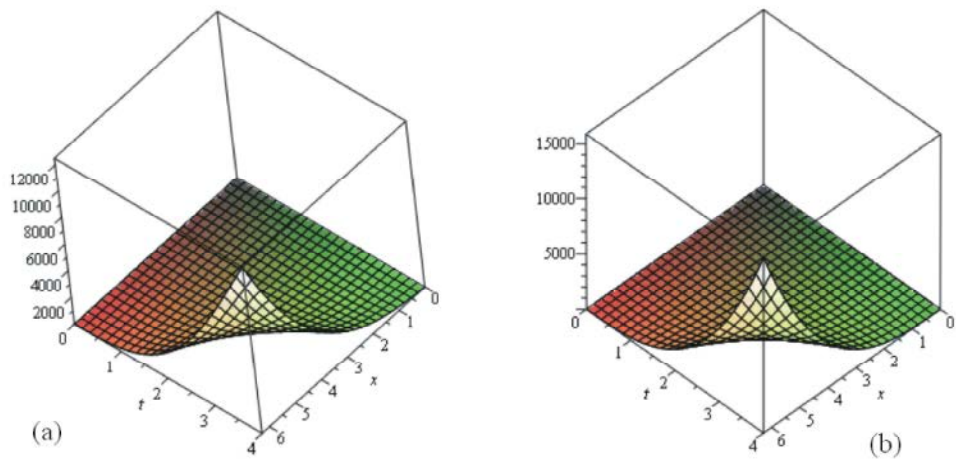


Fig. 3: (a), Approximate solution, (b), Exact solution

By using the Adomian decomposition method, we receive the following recurrence relation,

$$E[u_0(x,t)] = v^\alpha E[6x^3t + (x^3 - 6x)t^3],$$

$$E[u_{k+1}(x,t)] = v^\alpha E\left[\frac{\partial^2 u_k}{\partial x^2} - u_k\right], k \geq 0,$$

Accordingly,

$$u_0(x,t) = \frac{6x^3}{\Gamma(\alpha+2)}t^{\alpha+1} + \frac{6(x^3 - 6x)}{\Gamma(\alpha+4)}t^{\alpha+3},$$

$$u_1(x,t) = \frac{(36x - 6x^3)}{\Gamma(2\alpha+2)}t^{2\alpha+1} + \frac{(72x - 6x^3)}{\Gamma(2\alpha+4)}t^{2\alpha+3}.$$

$$u(x,t) = \frac{6x^3}{\Gamma(\alpha+2)}t^{\alpha+1} + \frac{6(x^3 - 6x)}{\Gamma(\alpha+4)}t^{\alpha+3} + \frac{(36x - 6x^3)}{\Gamma(2\alpha+2)}t^{2\alpha+1} + \frac{(72x - 6x^3)}{\Gamma(2\alpha+4)}t^{2\alpha+3} + \dots$$

Then, after coming down the noise terms, the closed form solution for $\alpha = 2$, is given as. $u(x,t) = x^3t^3$

CONCLUSIONS

Elzaki transform method is used to find appropriate resolutions of some one-dimensional wave equation: Burger's and Klein-Gordon fractional equations. It may be concluded that the proposed technique is very powerful and effective in finding analytic approximate solutions for a large class of partial differential equations of the fractional order. Numerical results explicitly reveal the utter reliability and efficiency of the proposed method.

REFERENCES

1. Duan, J.S. and M.Y. Xu, 2004. The problem for fractional diffusion-wave equations on finite interval and Laplace transform. *Appl. Math. J. Chin. Univ. Ser. A*, 19(2):165-171.
2. Gejji, V.D. and H. Jafari, 2007. Solving a multi-order fractional differential equation. *Appl. Math. Comput.*, 189: 541-548.
3. Hosseini, M.M., S.T. Mohyud-Din and A. Nakhaeei, 2011. New Rothe-Wavelet 7378 *Int. J. Phys. Sci. method for solving telegraph equations. Int. J. Syst. Sci. In Press.*
4. Jumarie, G., 2006. New stochastic fractional models for Malthusian growth, the Poissonian birth process and optimal management of populations. *Math. Comput. Model.*, 44: 231-254.
5. Jumarie, G., 2009. Laplace's transform of fractional order via the Mittag- Leffler function and modified Riemann-Liouville derivative. *Appl. Math. Lett.*, 22: 1659-1664.
6. Khan, Y. and N. Faraz, 2011. Modified fractional decomposition method having integral $(d\xi)^\alpha$, *J. King. Saud. Uni. Sci.*, 23: 157-161.
7. Miller, K.S and B. Ross, 1993. *An Introduction to the Fractional Calculus and Fractional Differential Equations*. Wiley, New York, pp: 384.
8. Mohyud-Din, S.T., M.A. Noor and K.I. Noor, 2009. Some relatively new techniques for nonlinear problems, *Mathematical Problems in Engineering*, Hindaw, 10: 1155-1180.
9. Mohyud-Din, S.T., M.A. Noor and K.I. Noor, 2009. Traveling wave solutions of seventh-order generalized KdV equation using He's polynomials. *Int. J. Nonlinear. Sci. Numer. Simul.*, 10: 227-233.
10. Momani, S. and M.A. Noor, 2006. Numerical methods for fourth-order fractional integro-differential equations. *Appl. Math. Comput.*, 182: 754-760.
11. Momani, S. and Z. Odibat, 2006. Analytical approach to linear fractional partial differential equations arising in fluid mechanics. *Phys. Lett. A.*, 355: 271-279.
12. Momani, S. and N.T. Shawagfeh, 2006. Decomposition method for solving fractional Riccati differential equations. *Appl. Math. Comput.*, 182: 1083-1092.

13. Oldham, K.B. and J. Spanier, 1974. The Fractional Calculus. Academic Press, New York, pp: 234.
14. Podlubny, I., 1999. Fractional Differential Equations. Academic Press, San Diego, pp: 368.
15. Ray, S.S., K.S. Chaudhuri and R.K. Bera, 2006. Analytical approximate solution of nonlinear dynamic system containing fractional derivative by modified decomposition method. Appl. Math. Comput., 182: 544-552.
16. Sweilam, N.H., M.M. Khader and R.F. Al-Bar, 2007. Numerical studies for a multi-order fractional differential equation. Phys. Lett. A., 371: 26-33.
17. Eman Hilal M.A. and M. Tarig, 2014. Elzaki, Solution of Nonlinear Partial Differential Equations by New Laplace Variational Iteration Method, Journal of Function Spaces, Volume 2014, pp: 1-5, <http://dx.doi.org/10.1155/2014/790714>.
18. Tarig M. Elzaki and J. Biazar, 2013. Homotopy Perturbation Method and Elzaki Transform for Solving System of Nonlinear Partial Differential Equations, World Applied Sciences Journal, (2013). DOI: 10.5829/idosi.wasj.2013.24.07.1041.
19. Tarig Elzaki, M., 2014. Application of Projected Differential Transform Method on Nonlinear Partial Differential Equations with Proportional Delay in One Variable, World Applied Sciences Journal, (2014). DOI: 10.5829/idosi.wasj.2014.30.03.1841.