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Modified Homotopy Perturbation Method by Using Sumudu Transform for Solving Initial Value Problems Represented By System of Nonlinear Partial Differential Equations

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Abstract: In this paper, we propose a new approximate solution, namely modified Homotopy Perturbation Sumudu transform method (MHPSTM) to handle various system of nonlinear partial differential equations. The modified Homotopy Perturbation Sumudu transform method is a combined form of the Homotopy Perturbation method and the Sumudu transform. We test our modified method on two examples and compare the results obtained of this modified method with Modified He Homotopy Perturbation Method (ETHPM). This modified method is very simple and the results are obtained very effective and fast.

Key words: Homotopy Perturbation Method, Sumudu Transform, Initial Value Problem, Partial differential equation

INTRODUCTION

The Homotopy Perturbation method was proposed first by He (2000) [1, 2] for solving linear and nonlinear Boundary Value Problems and Initial Value Problems. This method was applied to solve nonlinear partial differential equation [3]. The Sumudu transform was proposed by Watugala (1993) [4, 5]. The Sumudu transform operator was denoted by S[.] and it has been defined by the integral equation as:

$$S[f(t)] = G(v) = \frac{1}{v}\int_0^\infty e^{-\frac{t}{v}} f(t)\,dt, \qquad t \geq 0$$

where,

$$S\left[f^{(n)}(t)\right] = \frac{G(v)}{v^n} - \sum_{k=0}^{n-1} v^{-n+k} f^{(k)}(0), \qquad n \geq 1$$

The Objective of this paper is to apply the partial derivative on each partial differential equation of the system, and to combine the basic Homotopy Perturbation method with Sumudu transform.

The Comparisons of the results of proposed modified method (MHPSTM) on two applications with the Elzaki Transform Homotopy perturbation Method (ETHPM) [3] reveal that (MHPSTM) is very effective and convenient.

Modified Homotopy Perturbation Method by Sumudu Transform: Consider the following system of partial differential equation.

$$\begin{cases} \dfrac{\partial U_1}{\partial t} = g_1\left(x_1, \dots, x_{n-1}, t, U_1, \dots, U_n, \dfrac{\partial U_1}{\partial x_1}, \dots, \dfrac{\partial^2 U_1}{\partial^2 x_1}, \dots, \dfrac{\partial^p U_n}{\partial x_1 \dots \partial x_{n-1}}\right) \\ \vdots \\ \dfrac{\partial U_n}{\partial t} = g_n\left(x_1, \dots, x_{n-1}, t, U_1, \dots, U_n, \dfrac{\partial U_1}{\partial x_1}, \dots, \dfrac{\partial^2 U_1}{\partial^2 x_1}, \dots, \dfrac{\partial^p U_n}{\partial x_1 \dots \partial x_{n-1}}\right) \end{cases} \tag{1}$$

with initial conditions

$$\begin{cases} U_1(x_1, \dots, x_{n-1}, 0) = f_1(x_1, \dots, x_{n-1}) \\ \vdots \\ U_n(x_1, \dots, x_{n-1}, 0) = f_n(x_1, \dots, x_{n-1}) \end{cases}$$

where $g_1,\dots,g$ are nonlinear functions and $U_1,\dots,U_n$ are unknown functions.

Differentiating both sides of each equation of system (1), yields.

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$$\begin{cases} \dfrac{\partial^2 U_1}{\partial t^2} = g_1\left(x_1, \dots, x_{n-1}, t, \dfrac{\partial U_1}{\partial t}, \dots, \dfrac{\partial^2 U_1}{\partial t \partial x_1}, \dots, \dfrac{\partial^3 U_1}{\partial t \partial^2 x_1}, \dots, \dfrac{\partial^{p+1} U_n}{\partial t \partial x_1 \dots \partial x_{n-1}} \right) \\ \vdots \\ \dfrac{\partial^2 U_n}{\partial t^2} = g_n\left(x_1, \dots, x_{n-1}, t, \dfrac{\partial U_1}{\partial t}, \dots, \dfrac{\partial^2 U_1}{\partial t \partial x_1}, \dots, \dfrac{\partial^3 U_1}{\partial t \partial^2 x_1}, \dots, \dfrac{\partial^{p+1} U_n}{\partial t \partial x_1 \dots \partial x_{n-1}} \right) \end{cases} \tag{2}$$

Using system (1), we get,

$$\begin{cases} \dfrac{\partial U_1(x_1, \dots, x_{n-1}, 0)}{\partial t} = g_1\left(x_1, \dots, x_{n-1}, 0, U_1, \dots, U_n, \dfrac{\partial U_1}{\partial x_1}, \dots, \dfrac{\partial^2 U_1}{\partial^2 x_1}, \dots, \dfrac{\partial^p U_n}{\partial x_1 \dots \partial x_{n-1}} \right) \\ \qquad = K_1(x_1, \dots, x_{n-1}) \\ \vdots \\ \dfrac{\partial U_n(x_1, \dots, x_{n-1}, 0)}{\partial t} = g_n\left(x_1, \dots, x_{n-1}, 0, U_1, \dots, U_n, \dfrac{\partial U_1}{\partial x_1}, \dots, \dfrac{\partial^2 U_1}{\partial^2 x_1}, \dots, \dfrac{\partial^p U_n}{\partial x_1 \dots \partial x_{n-1}} \right) \\ \qquad = K_n(x_1, \dots, x_{n-1}) \end{cases}$$

Taking the Sumudu Transform on system (2), yields

$$\begin{cases} \dfrac{S(U_1)}{v^2} - \dfrac{1}{v^2} U_1(x_1, \dots, x_{n-1}, 0) - \dfrac{1}{v} \dfrac{\partial U_1(x_1, \dots, x_{n-1}, 0)}{\partial t} \\ \quad = S\left[g_1\left(x_1, \dots, x_{n-1}, t, \dfrac{\partial U_1}{\partial t}, \dots, \dfrac{\partial^2 U_1}{\partial t \partial x_1}, \dots, \dfrac{\partial^3 U_1}{\partial t \partial^2 x_1}, \dots, \dfrac{\partial^{p+1} U_n}{\partial t \partial x_1 \dots \partial x_{n-1}} \right) \right] \\ \quad \vdots \\ \dfrac{S(U_n)}{v^2} - \dfrac{1}{v^2} U_n(x_1, \dots, x_{n-1}, 0) - \dfrac{1}{v} \dfrac{\partial U_n(x_1, \dots, x_{n-1}, 0)}{\partial t} \\ \quad = S\left[g_n\left(x_1, \dots, x_{n-1}, t, \dfrac{\partial U_1}{\partial t}, \dots, \dfrac{\partial^2 U_1}{\partial t \partial x_1}, \dots, \dfrac{\partial^3 U_1}{\partial t \partial^2 x_1}, \dots, \dfrac{\partial^{p+1} U_n}{\partial t \partial x_1 \dots \partial x_{n-1}} \right) \right] \end{cases}$$

Then,

$$\begin{cases} S(U_1) = f_1(x_1, \dots, x_{n-1}) + K_1(x_1, \dots, x_{n-1})\ v \\ \quad + v^2 S\left[g_1\left(x_1, \dots, x_{n-1}, t, \dfrac{\partial U_1}{\partial t}, \dots, \dfrac{\partial^2 U_1}{\partial t \partial x_1}, \dots, \dfrac{\partial^3 U_1}{\partial t \partial^2 x_1}, \dots, \dfrac{\partial^{p+1} U_n}{\partial t \partial x_1 \dots \partial x_{n-1}} \right) \right] \\ \quad \vdots \\ S(U_n) = f_n(x_1, \dots, x_{n-1}) + K_n(x_1, \dots, x_{n-1})\ v \\ \quad + v^2 S\left[g_n\left(x_1, \dots, x_{n-1}, t, \dfrac{\partial U_1}{\partial t}, \dots, \dfrac{\partial^2 U_1}{\partial t \partial x_1}, \dots, \dfrac{\partial^3 U_1}{\partial t \partial^2 x_1}, \dots, \dfrac{\partial^{p+1} U_n}{\partial t \partial x_1 \dots \partial x_{n-1}} \right) \right] \end{cases} \tag{3}$$

The homotopy of system (3) can be written as follows;

$$\begin{cases} S(U_1) = f_1(x_1,\dots,x_{n-1}) + K_1(x_1,\dots,x_{n-1})\ v \\ \qquad + pv^2 S\left[g_1\left(x_1,\dots,x_{n-1},t,\dfrac{\partial U_1}{\partial t},\dots,\dfrac{\partial^2 U_1}{\partial t\partial x_1},\dots,\dfrac{\partial^3 U_1}{\partial t\partial^2 x_1},\dots,\dfrac{\partial^{p+1} U_n}{\partial t\partial x_1\dots\partial x_{n-1}} \right)\right] \\ \quad\vdots \\ S(U_n) = f_n(x_1,\dots,x_{n-1}) + K_n(x_1,\dots,x_{n-1})\ v \\ \qquad + pv^2 S\left[g_n\left(x_1,\dots,x_{n-1},t,\dfrac{\partial U_1}{\partial t},\dots,\dfrac{\partial^2 U_1}{\partial t\partial x_1},\dots,\dfrac{\partial^3 U_1}{\partial t\partial^2 x_1},\dots,\dfrac{\partial^{p+1} U_n}{\partial t\partial x_1\dots\partial x_{n-1}} \right)\right] \end{cases} \tag{4}$$

where $p \in [0,1]$ is an embedding parameter.

According to the HHPM the solution of system (4) can be written as a power series in p

$$\begin{cases} U_1 = \sum_{i=0}^{\infty} p^i U_{1i} \\ \vdots \\ U_n = \sum_{i=0}^{\infty} p^i U_{ni} \end{cases} \tag{5}$$

Substituting system (5) into system (4), and comparing coefficients of terms with identical powers of p in the result system, leads to.

$$p^0 : \begin{cases} S(U_1) = f_1(x_1,\dots,x_{n-1}) + K_1(x_1,\dots,x_{n-1})\ v \\ \vdots \\ S(U_n) = f_n(x_1,\dots,x_{n-1}) + K_n(x_1,\dots,x_{n-1})\ v \end{cases}$$

$$p^{i+1} : \begin{cases} S(U_{1i+1}) = v^2 S\left[g_1\left(\begin{matrix} x_1,\dots,x_{n-1},t,\dfrac{\partial U_1}{\partial t},\dots,\dfrac{\partial^2 U_1}{\partial t\partial x_1} \\ ,\dots,\dfrac{\partial^3 U_1}{\partial t\partial^2 x_1},\dots,\dfrac{\partial^{p+1} U_n}{\partial t\partial x_1\dots\partial x_{n-1}} \end{matrix} \right)\right] \\ \quad\vdots \\ S(U_{ni+1}) = v^2 S\left[g_n\left(\begin{matrix} x_1,\dots,x_{n-1},t,\dfrac{\partial U_1}{\partial t},\dots,\dfrac{\partial^2 U_1}{\partial t\partial x_1} \\ ,\dots,\dfrac{\partial^3 U_1}{\partial t\partial^2 x_1},\dots,\dfrac{\partial^{p+1} U_n}{\partial t\partial x_1\dots\partial x_{n-1}} \end{matrix} \right)\right] \end{cases}$$

where $i = 0,1,2,...$

Taking the inverse Sumudu Transform, gives,

$$U_{1i} \quad , \quad U_{ni} \quad ; \quad i = 0,1,2,...$$

Setting $p = 1$, we have the approximate solution of System (1)

$$\begin{cases} U_1 = \sum_{i=0}^{\infty} U_{1i} \\ \vdots \\ U_n = \sum_{i=0}^{\infty} U_{ni} \end{cases}$$

Illustrative Examples: Now we apply the modified method (MHPSTM) to solve the following examples

Example 1: Consider the system of nonlinear partial differential equations

$$\begin{cases} U_t = U_{xx} + 2UU_x - (UW)_x \\ W_t = W_{xx} + 2WW_x - (UW)_x \end{cases} ; U = U(x,t) \quad , \quad W = W(x,t) \tag{6}$$

With initial conditions,

$$U(x,0) = \sin(x) \quad , \quad W(x,0) = \sin(x)$$

Differentiating both sides of system (6), yields

$$\begin{cases} U_{tt} = U_{xxt} + 2U_tU_x + 2UU_{xt} - (UW)_{xt} \\ W_{tt} = W_{xxt} + 2W_tW_x + 2WW_{xt} - (UW)_{xt} \end{cases} \tag{7}$$

Using system (6), we have new initial conditions,

$$U_t(x,0) = -\sin(x) \quad , \quad W_t(x,0) = -\sin(x)$$

Applying the Sumudu Transform on system (7), yields,

$$\begin{cases} \dfrac{S(U)}{v^2} - \dfrac{1}{v^2}U(x,0) - \dfrac{1}{v}U_t(x,0) \\ \quad = S\left[U_{xxt} + 2U_tU_x + 2UU_{xt} - (UW)_{xt}\right] \\ \dfrac{S(W)}{v^2} - \dfrac{1}{v^2}W(x,0) - \dfrac{1}{v}W_t(x,0) \\ \quad = S\left[W_{xxt} + 2W_tW_x + 2WW_{xt} - (UW)_{xt}\right] \end{cases}$$

Then we have,

$$\begin{cases} S(U) = \sin(x) - (\sin(x))v \\ \quad + v^2 S\left[U_{xxt} + 2U_tU_x + 2UU_{xt} - (UW)_{xt}\right] \\ S(W) = \sin(x) - (\sin(x))v \\ \quad + v^2 S\left[W_{xxt} + 2W_tW_x + 2WW_{xt} - (UW)_{xt}\right] \end{cases} \tag{8}$$

Now, constructing the homotopy on system (8) as follows,

$$\begin{cases} S(U) = \sin(x) - (\sin(x))v \\ \quad + pv^2 S\left[U_{xxt} + 2U_tU_x + 2UU_{xt} - (UW)_{xt}\right] \\ S(W) = \sin(x) - (\sin(x))v \\ \quad + pv^2 S\left[W_{xxt} + 2W_tW_x + 2WW_{xt} - (UW)_{xt}\right] \end{cases} \tag{9}$$

Substituting system (5) into system (9), and comparing coefficients of terms with identical powers of p in the result system, leads to.

$$p^0 : \begin{cases} S(U_0) = \sin(x) - (\sin(x))v \\ S(W_0) = \sin(x) - (\sin(x))v \end{cases} \tag{10}$$

$$p^1 : \begin{cases} S(U_1) = v^2 S\left[U_{0xxt} + 2U_{0t}U_{0x} + 2U_0U_{0xt} - (U_0W_0)_{xt}\right] \\ S(W_0) = v^2 S\left[W_{0xxt} + 2W_{0t}W_{0x} + 2W_0W_{0xt} - (U_0W_0)_{xt}\right] \end{cases}$$
$$\vdots \tag{11}$$

Taking the inverse Sumudu Transform of system (10), system (11) and system …, gives,

$$p^0 : \begin{cases} U_0(x,t) = \sin(x) - (\sin(x))t \\ W_0(x,t) = \sin(x) - (\sin(x))t \end{cases}$$

$$p^1 : \begin{cases} U_1(x,t) = (\sin(x))\dfrac{t^2}{2!} \\ W_1(x,t) = (\sin(x))\dfrac{t^2}{2!} \end{cases}$$

$$p^2 : \begin{cases} U_2(x,t) = -(\sin(x))\dfrac{t^3}{3!} \\ W_2(x,t) = -(\sin(x))\dfrac{t^3}{3!} \end{cases}$$

Then we obtain the solutions of system (6) as follow;

$$\begin{cases} U(x,t) = U_0(x,t) + U_1(x,t) + U_2(x,t) + ... \\ \quad = \sin(x) - (\sin(x))v + (\sin(x))\dfrac{t^2}{2!} - (\sin(x))\dfrac{t^3}{3!} + ... \\ \quad = \sin(x)\, e^{-t} \\ W(x,t) = W_0(x,t) + W_1(x,t) + W_2(x,t) + ... \\ \quad = \sin(x) - (\sin(x))v + (\sin(x))\dfrac{t^2}{2!} - (\sin(x))\dfrac{t^3}{3!} + ... \\ \quad = \sin(x)\, e^{-t} \end{cases}$$

Example 2: Consider the system of nonlinear partial differential equations.

$$\begin{cases} U_t = -\psi_x W_y + \psi_y W_x - U & U = U(x,y,t) \\ \psi_t = -W_x U_y - W_y U_x + \psi & ; \quad \psi = \psi(x,y,t) \\ W_t = -U_x \psi_y - U_y \psi_x + W & W = W(x,y,t) \end{cases} \tag{12}$$

With initial conditions,

$U(x,y,0)=e^{x+y},\quad \psi(x,y,0)=e^{x-y},\quad W(x,y,0)=e^{-x+y}$

Differentiating both sides of system (12), yeilds

$$\begin{cases} U_{tt}=-\psi_{xt}W_y-\psi_x W_{yt}+\psi_{yt}W_x+\psi_y W_{xt}-U_t \\ \psi_{tt}=-W_{xt}U_y-W_x U_{yt}-W_{yt}U_x-W_y U_{xt}+\psi_t \\ W_{tt}=-U_{xt}\psi_y-U_x\psi_{yt}-U_{yt}\psi_x-U_y\psi_{xt}+W_t \end{cases} \tag{13}$$

Using system (12), we have new initial conditions,

$U_t(x,y,0)=-e^{x+y},\quad \psi_t(x,y,0)=e^{x-y},\quad W_t(x,y,0)=e^{-x+y}$

Taking the Sumudu Transform on system (13), yields,

$$\begin{cases} \dfrac{S(U)}{v^2}-\dfrac{1}{v^2}U(x,y,0)-\dfrac{1}{v}U_t(x,y,0) \\ \quad =S\left[-\psi_{xt}W_y-\psi_x W_{yt}+\psi_{yt}W_x+\psi_y W_{xt}-U_t\right] \\ \dfrac{S(\psi)}{v^2}-\dfrac{1}{v^2}\psi(x,y,0)-\dfrac{1}{v}\psi_t(x,y,0) \\ \quad =S\left[-W_{xt}U_y-W_x U_{yt}-W_{yt}U_x-W_y U_{xt}+\psi_t\right] \\ \dfrac{S(W)}{v^2}-\dfrac{1}{v^2}W(x,y,0)-\dfrac{1}{v}W_t(x,y,0) \\ \quad =S\left[-U_{xt}\psi_y-U_x\psi_{yt}-U_{yt}\psi_x-U_y\psi_{xt}+W_t\right] \end{cases}$$

Then we have,

$$\begin{cases} S(U)=e^{x+y}-e^{x+y}v \\ \quad +v^2S\left[-\psi_{xt}W_y-\psi_x W_{yt}+\psi_{yt}W_x+\psi_y W_{xt}-U_t\right] \\ S(\psi)=e^{x-y}+e^{x-y}v \\ \quad +v^2S\left[-W_{xt}U_y-W_x U_{yt}-W_{yt}U_x-W_y U_{xt}+\psi_t\right] \\ S(W)=e^{-x+y}+e^{-x+y}v \\ \quad +v^2S\left[-U_{xt}\psi_y-U_x\psi_{yt}-U_{yt}\psi_x-U_y\psi_{xt}+W_t\right] \end{cases} \tag{14}$$

Now, constructing the homotopy on system (14) as follows,

$$\begin{cases} S(U)=e^{x+y}-e^{x+y}v \\ \quad +pv^2S\left[-\psi_{xt}W_y-\psi_x W_{yt}+\psi_{yt}W_x+\psi_y W_{xt}-U_t\right] \\ S(\psi)=e^{x-y}+e^{x-y}v \\ \quad +pv^2S\left[-W_{xt}U_y-W_x U_{yt}-W_{yt}U_x-W_y U_{xt}+\psi_t\right] \\ S(W)=e^{-x+y}+e^{-x+y}v \\ \quad +pv^2S\left[-U_{xt}\psi_y-U_x\psi_{yt}-U_{yt}\psi_x-U_y\psi_{xt}+W_t\right] \end{cases} \tag{15}$$

Substituting system (5) into system (15), and comparing coefficients of terms with identical powers of p in the result system, leads to,

$$p^0:\begin{cases} S(U_0)=e^{x+y}-e^{x+y}v \\ S(\psi_0)=e^{x-y}+e^{x-y}v \\ S(W_0)=e^{-x+y}+e^{-x+y}v \end{cases} \tag{16}$$

$$p^1:\begin{cases} S(U_1)=v^2S\left[-\psi_{0xt}W_{0y}-\psi_{0x}W_{0yt}+\psi_{0yt}W_{0x}+\psi_{0y}W_{0xt}-U_{0t}\right] \\ S(\psi_1)=v^2S\left[-W_{0xt}U_{0y}-W_{0x}U_{0yt}-W_{0yt}U_{0x}-W_{0y}U_{0xt}+\psi_{0t}\right] \\ S(W_1)=v^2S\left[-U_{0xt}\psi_{0y}-U_{0x}\psi_{0yt}-U_{0yt}\psi_{0x}-U_{0y}\psi_{0xt}+W_{0t}\right] \end{cases}$$

$\vdots$

$$\tag{17}$$

Taking the inverse Sumudu Transform of system (16), system (17) and system …, yields,

$$p^0:\begin{cases} U_0(x,y,t)=e^{x+y}-e^{x+y}t \\ \psi_0(x,y,t)=e^{x-y}+e^{x-y}t \\ W_0(x,y,t)=e^{-x+y}+e^{-x+y}t \end{cases}$$

$$p^1:\begin{cases} U_1(x,y,t)=e^{x+y}\dfrac{t^2}{2!} \\ \psi_1(x,y,t)=e^{x-y}\dfrac{t^2}{2!} \\ W_1(x,y,t)=e^{-x+y}\dfrac{t^2}{2!} \end{cases}$$

$$p^2:\begin{cases} U_2(x,y,t)=-e^{x+y}\dfrac{t^3}{3!} \\ \psi_2(x,y,t)=e^{x-y}\dfrac{t^3}{3!} \\ W_1(x,y,t)=e^{-x+y}\dfrac{t^3}{3!} \end{cases}$$

Then we obtain the solution of system (12) as follow;

CONCLUSION

In this paper, Modified Homotopy Perturbation Sumudu transform method (MHPSTM) has been implemented to find the solution of various kinds of system of nonlinear partial differential equations. The comparisons of our results with the ETHPM, clearly show that U_0, ψ_0, W_0 by (MHPSTM) better than U_0, ψ_0, W_0 by (ETHPM) and U_0+U_1, $\psi_0+\psi_1$, W_0+W_1 by (MHPSTM) better than U_0+U_1, $\psi_0+\psi_1$, W_0+W_1 by (ETHPM).

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