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Bayes and Parametric Bootstrap Methods for Estimating the Parameters of Distributions Having Power Hazard Function

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Abstract: This article deals with the statistical inference for a step-stress partially accelerated life tests with two stress levels under progressive type-II censoring. The lifetime of the test units is assumed to follow distributions having power hazard function (DPHF). The maximum likelihood (ML), Bayes and parametric bootstrap methods are used for estimating unknown parameters of DPHF and the acceleration factor. Based on normal approximation to the asymptotic distribution of MLEs, the approximate confidence intervals for the parameters and the acceleration factor are derived. In addition, two bootstrap confidence intervals are also proposed. The classical Bayes estimates cannot be obtained in explicit form, so we propose to apply the Markov chain Monte Carlo (MCMC) method to tackle this problem, which allows us to construct the credible interval of the involved parameters. Finally, analysis of a simulated data set has also been presented to illustrate the proposed estimation methods.

Key words: Distributions having power hazard function · Step-stress partially accelerated life test model · Bootstrap methods · Bayesian estimation · MCMC method

INTRODUCTION

Due to the continual improvement in the manufacturing design, it is more difficult to obtain information about the lifetime of products or materials with high reliability at the time of testing under normal conditions. These make the lifetime testing under normal conditions very costly and take a long time. For this reason, accelerated life test (ALT) and partially accelerated life test (PALT) the most common approaches that are used in order to obtain failures quickly, in a short period of time. Accelerated life testing is achieved by subjecting the test units to conditions that are more severe than the normal ones, such as higher levels of temperature, voltage, pressure, vibration, load, etc. Units are tested at high stress levels to induce early failures and then the failure information is related to that at an operational stress level through a given stress- dependent model. When such model is unknown, the accelerated life test cannot be conducted and instead the partially accelerated life tests (PALT) become suitable. The PALT combines both ordinary and accelerated life tests. The aim of such testing is to rapid obtaining data, which yield desired information on product life or performance under normal use. Thus, PALT is used for reliability analysis to save more time and money over the ordinary life tests. PALT can be carried out using constant-stress, step-stress, or progressive-stress. According to Nelson [1] and Rao [2], the common methods are step-stress and constant-stress. Under step-stress PALT (SSPALT), a test units first run at normal (use) condition and, if it does not fail for a specified time, then it is run at accelerated condition until the test terminates. But the constant-stress PALT runs each unit at either use condition or accelerated condition only, i.e. each unit is run at a constant-stress level until it fails or censors. Several authors have dealt with this type of PALT, including DeGroot and Goel [3], Bai and Chung [4], Ismail [5], Abdel-Hamid and Al-Hussaini [6], Bai *et al*. [7], Abdel-Ghani [8], Abd-Elfattah [9], Bhattacharyya and Soejoeti [10], Ismail and Aly [11] and EL-Sagheer [12].

In life testing and reliability studies, the experimenter may not always obtain complete information on failure times for all experimental units. Data obtained from such experiments are called censored data. Saving the total time on test and the cost associated with it are some of the major reasons for censoring. A censoring scheme (CS), which can balance between total time spent for the

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experiment, number of units used in the experiment and the efficiency of statistical inference based on the results of the experiment, is desirable. The most common CSs are type-I (time) censoring and type-II (unit) censoring. The conventional type-I and type-II CSs do not have the flexibility of allowing removal of units at points other than the terminal point of the experiment. Because of that, a more general CS called progressive type-II censoring has been used in this article. This type of CS allows to the experimenter to save time and cost and it is useful when the units being tested are very expensive. Schematically a progressively type-II censored sample can be described as follows. Suppose that n independent units are put on a life test with continuous identically distributed failure times X_1, X_2, ..., X_n. Suppose further that a censoring scheme (R_1, R_2, ..., R_m) is previously fixed such that immediately following the first failure X_1, R_1 surviving units are removed from the experiment at random and immediately following the second failure X_2, R_2 surviving items are removed from the experiment at random. This process continues until, at the time of the *m-th* observed failure x_m, the remaining R_m surviving units are removed from the test. The m ordered observed failure times denoted by $X_{1:m:n}^{(R_1,\ldots,R_m)}$, $X_{2:m:n}^{(R_1,\ldots,R_m)}, \ldots, X_{m:m:n}^{(R_1,\ldots,R_m)}$ are called progressively type-II right censored order statistics of size m from a sample of size with progressive censoring scheme (R_1, R_2,....,R_m). It is clear that $n = m + \sum_{i=1}^{m} R_i$. The special case when $R_1 = R_2 = \cdots = R_{m-1} = 0$ so that $R_m = n - m$ is the case of conventional type-II right censored sampling. Also when $R_1 = R_2 = \cdots = R_m = 0$, so that $m = n$, the progressively type-II right censoring scheme reduces to the case of no censoring (ordinary order statistics) see Balakrishnan and Aggarwala [13], Balakrishnan [14] and EL-Sagheer [15].

This article interested in the estimation of the parameters and the acceleration factor when the sample is available progressive type-II censoring scheme from distribution has a power hazard function under SSPALT model. The power hazard function has been defined by Mugdadi [16] as:

$$h(x) = \alpha t^{\gamma-1} \qquad ,\alpha > 0, \gamma > 0, t > 0. \tag{1}$$

Corresponding to this hazard function, cumulative distribution function (cdf), probability density function (pdf) and survival function are given respectively by

$$F(t;\alpha,\gamma) = 1 - \exp\left\{-\frac{\alpha}{\gamma} t^{\gamma}\right\} \qquad ,\alpha > 0, \gamma > 0, t > 0, \tag{2}$$

$$f(t;\alpha,\gamma) = \alpha t^{\gamma-1} \exp\left\{-\frac{\alpha}{\gamma} t^{\gamma}\right\} \qquad ,\alpha > 0, \gamma > 0, t > 0, \tag{3}$$

and

$$S(t) = \exp\left\{-\frac{\alpha}{\gamma} t^{\gamma}\right\} \qquad ,\alpha > 0, \gamma > 0, t > 0. \tag{4}$$

In the sequel, distribution with density defined in (2) will be referred to the distribution has a power hazard function (DPHF for brevity). For $0 < \gamma < 1$, the DPHF has a decreasing hazard function. For $\gamma > 1$, the DPHF has an increasing hazard function. It is clear that some well-known life time distributions as the Weibull, Rayleigh and exponential is special cases of the DPHF distribution. Such that,

If $\gamma = \alpha$ then DPHF reduces to *Weibull* (α,1)
If $\alpha = \frac{1}{\lambda^2}, \gamma = 2$ then DPHF reduces to *Rayleigh*(γ)
If $\gamma = 1$ then DPHF reduces to exponential distribution with mean $1/\alpha$.

Therefore, the results obtained in our study will be valid for weibull, Rayleigh, exponential distributions and the other distributions have power hazard function.

The rest of this paper is organized as follows. In Section 2 we provide some basic assumptions and test procedure. Section 3, presents the derivation of the maximum likelihood estimators (MLEs) of the involved parameters as well as the corresponding approximate confidence intervals (ACIs). The two parametric bootstrap confidence intervals (CIs) for the parameters are discussed in Section 4. Section 5 deals with the Bayesian approach that uses the well-known Markov chain Monte Carlo method. A simulation example to illustrate the approach is given in Section 6. Finally, Section 7 provides some concluding remarks.

MATERIALS AND METHODS

Basic Assumptions and Test Procedure: The following assumptions are used throughout the paper in the framework of step-stress partially accelerated life test based on progressively type-II censored sample:

- n Identical and independent units are put on the life test.
- The lifetime of each unit has PHFD (α, γ).
- The test is terminated at m-th failure, where m is prefixed ($m \leq n$).

- Each of the n units is first run under normal use condition. If it does not fail or remove from the test by a pre-specified time τ, it is put under accelerated condition (stress).
- At the i-th failure a random number of the surviving units, R_i, $i = 1,2,...,m-1$ are randomly selected and removed from the test. Finally, at m-th failure the remaining surviving units $R_m = n - m - \sum_{i=1}^{m-1} R_i$ are all removed from the test and the test is terminated.
- Let n_1 be the number of failures before time at the normal condition and let $m - n_1$ be the number of failures after time τ at accelerated condition, then, the observed progressive censored data are

$$y_{1;m,n}^{R} < \ldots < y_{n_1;m,n}^{R} < \tau < y_{n_1+1;m,n}^{R} < \ldots < y_{m;m,n}^{R},$$

where $R = (R_1, R_2, ..., R_m)$ and $\sum_{i=1}^{m} R_i = n - m$.

- According to DeGroot and Goel [3] the lifetime denoted by Y, of a unit under SSPALT can be written as:

$$Y = \begin{cases} T & \text{if } T \le \tau, \\ \tau + \theta^{-1}(T - \tau) & \text{if } T > \tau, \end{cases} \tag{6}$$

where T is the lifetime of the units under normal condition, τ is the stress change time and θ is the acceleration factor ($\theta > 1$).

From (6), the probability density function of a total lifetime Y of test unit can be written as;

$$f(y) = \begin{cases} 0, & y \le 0, \\ f_1(y) = \alpha y^{\gamma-1} \exp\left\{-\dfrac{\alpha}{\gamma} y^{\gamma}\right\}, & 0 < y \le \tau, \\ f_2(y) = \alpha\theta\left[\tau + \theta(y - \tau)\right]^{\gamma-1} \exp\left\{-\dfrac{\alpha}{\gamma}\left[\tau + \theta(y - \tau)\right]^{\gamma}\right\}, & y > \tau, \end{cases} \tag{7}$$

which is obtained by the transformation variable technique using the density in (3) and the model proposed by DeGroot and Goel [3] which is given in (6).

Maximum Likelihood Inference: In this Section, the point and interval estimations of the model parameters and acceleration factor are introduced using the maximum likelihood method based on progressive type-II censoring. Also, Fisher information matrix of the model parameters and acceleration factor are presented.

Point Estimation: Let $y_i = y_{i;m,n}^{R}, i = 1,2,\ldots,m,$ be the observed values of the lifetime Y obtained from a progressive censoring scheme under SSPALT with censored scheme $R = (R_1, R_2, ..., R_m)$. The maximum likelihood function of the observations $y_{1;m,n}^{R} < \ldots < y_{n_1;m,n}^{R} < \tau < y_{n_1+1;m,n}^{R} < \ldots < y_{m;m,n}^{R},$ is given by;

$$L(\alpha, \gamma, \theta \mid \underline{y}) = C\left\{\prod_{i=1}^{n_1} f_1(y_i)\left[S_1(y_i)\right]^{R_i}\right\}\left\{\prod_{i=n_1+1}^{m} f_2(y_i)\left[S_2(y_i)\right]^{R_i}\right\}, \tag{8}$$

where $C = n(n-1-R_1)(n-2-R_1-R_2)\ldots\left(n-m+1-\sum_{i=1}^{m-1} R_i\right),$ and

$$\begin{cases} S_1(y) = \exp\left\{-\dfrac{\alpha}{\gamma} y^{\gamma}\right\}, \\ \\ S_2(y) = \exp\left\{-\dfrac{\alpha}{\gamma}\left[\tau + \theta(y-\tau)\right]^{\gamma}\right\}. \end{cases}, \tag{9}$$

From (7)-(9), we get

$$L(\alpha,\gamma,\theta \mid \underline{y}) = C\ \alpha^m\ \theta^{(m-n_1)}\ \exp\left\{-\frac{\alpha}{\gamma}\left[\sum_{i=1}^{n_1}(R_i+1)y_i^{\gamma} + \sum_{i=n_1+1}^{m}(R_i+1)(\phi_i(\theta))^{\gamma}\right]\right\} \tag{10}$$
$$\times\left[\prod_{i=1}^{n_1} y_i^{(\gamma-1)}\right]\left[\prod_{i=n_1+1}^{m}(\phi_i(\theta))^{(\gamma-1)}\right],$$

where,

$$\phi_i(\theta) = \tau + \lambda(y_i - \tau). \tag{11}$$

Therefore, the natural logarithm of the likelihood function $\ell(\alpha,\gamma,\theta \mid \underline{y}) = \log L(\alpha,\gamma,\theta \mid \underline{y})$ without normalized constant is then given by;

$$\ell(\alpha,\gamma,\theta \mid \underline{y}) = m\log\alpha + (m-n_1)\log\theta + (\gamma-1)\left[\sum_{i=1}^{n_1}\log y_i + \sum_{i=n_1+1}^{m}\log\phi_i(\theta)\right]$$
$$-\frac{\alpha}{\gamma}\left[\sum_{i=1}^{n_1}(R_i+1)y_i^{\gamma} + \sum_{i=n_1+1}^{m}(R_i+1)(\phi_i(\theta))^{\gamma}\right]. \tag{12}$$

Calculating the first partial derivatives of Equation (12) with respect to α, γ and θ and equating each to zero, we get the likelihood equations as

$$\frac{\partial\ell(\alpha,\gamma,\theta \mid \underline{y})}{\partial\alpha} = \frac{m}{\alpha} - \frac{1}{\gamma}\left[\sum_{i=1}^{n_1}(R_i+1)y_i^{\gamma} + \sum_{i=n_1+1}^{m}(R_i+1)(\phi_i(\theta))^{\gamma}\right] = 0. \tag{13}$$

Therefore,

$$\hat{\alpha}(\gamma,\theta) = m\left[\frac{1}{\gamma}\left(\sum_{i=1}^{n_1}(R_i+1)y_i^{\gamma} + \sum_{i=n_1+1}^{m}(R_i+1)(\phi_i(\theta))^{\gamma}\right)\right]^{-1}, \tag{14}$$

$$\frac{\partial\ell(\alpha,\gamma,\theta \mid \underline{y})}{\partial\gamma} = \sum_{i=1}^{n_1}\log y_i + \sum_{i=n_1+1}^{m}\log\phi_i(\theta)$$
$$-\alpha\left[\frac{1}{\gamma}\left(\sum_{i=1}^{n_1}(R_i+1)y_i^{\gamma}\log y_i + \sum_{i=n_1+1}^{m}(R_i+1)(\phi_i(\theta))^{\gamma}\log\phi_i(\theta)\right)\right.$$
$$\left.+\frac{1}{\gamma^2}\left(\sum_{i=1}^{n_1}(R_i+1)y_i^{\gamma} + \sum_{i=n_1+1}^{m}(R_i+1)(\phi_i(\theta))^{\gamma}\right)\right] = 0, \tag{15}$$

And

$$\frac{\partial\ell(\alpha,\gamma,\theta\mid \underline{y})}{\partial\theta}=\frac{m-n_1}{\theta}-\alpha\sum_{i=n_1+1}^{m}(R_i+1)(y_i-\tau)\left[\phi_i(\theta)\right]^{(\gamma-1)}+(\gamma-1)\sum_{i=n_1+1}^{m}\frac{(y_i-\tau)}{\phi_i(\theta)}=0. \tag{16}$$

Now, we have a system of three non-linear likelihood equations (14)-(16) in three unknowns α, γ and θ. It cannot be solved analytically. The Newton-Raphson iteration method is used to obtain the estimates. The algorithm is described as follows:

- Use the method of moments or any other methods to estimate the parameters α, γ and θ as starting point of iteration, denote the estimates as (α_0, γ_0, θ_0) and set $k = 0$.
- Calculate $\left(\frac{\partial\ell(\alpha,\gamma,\theta\mid \underline{y})}{\partial\alpha},\frac{\partial\ell(\alpha,\gamma,\theta\mid \underline{y})}{\partial\gamma},\frac{\partial\ell(\alpha,\gamma,\theta\mid \underline{y})}{\partial\theta}\right)_{(\alpha_k,\gamma_k,\theta_k)}$ and the observed Fisher Information matrix $I^{-1}(\alpha,\gamma,\theta)$, given in the next subsection.
- Update (α, γ, θ) as.

$$(\alpha_{k+1},\gamma_{k+1},\theta_{k+1})=(\alpha_k,\gamma_k,\theta_k)+\left(\frac{\partial\ell}{\partial\alpha},\frac{\partial\ell}{\partial\gamma},\frac{\partial\ell}{\partial\theta}\right)_{(\alpha_k,\gamma_k,\theta_k)}\times I^{-1}(\alpha,\gamma,\theta). \tag{17}$$

- Set $k = k + 1$ and then go back to Step 1.
- Continue the iterative steps until $\left|(\alpha_{k+1},\gamma_{k+1},\theta_{k+1})-(\alpha_k,\gamma_k,\theta_k)\right|$ is smaller than a threshold value. The final estimates of (α, γ, θ) are the MLE of the parameters, denoted as $(\hat{\alpha},\hat{\gamma},\hat{\theta})$.

Approximate Confidence Intervals: As indicated by Vander and Meeker [17] the most common method to set confidence bounds for the parameters is to use the asymptotic normal distribution of the MLEs. The asymptotic variance and covariance of the MLEs, $\hat{\alpha},\hat{\gamma}$, and $\hat{\theta}$ are given by the entries of the inverse of the Fisher information matrix $I_{ij}=E\left[-\partial^2\ell(\Phi)/\partial\eta_i\partial\eta_j\right]$ where i,j = 1,2,3 and $\Phi=(\eta_1,\eta_2,\eta_3)=(\alpha,\gamma,\theta)$. Unfortunately, the exact closed forms for the above expectations are difficult to obtain. Therefore, the observed Fisher information matrix $\hat{I}_{ij}=E\left[-\partial^2\ell(\Phi)/\partial\eta_i\partial\eta_j\right]_{\Phi=\hat{\Phi}}$, which is obtained by dropping the expectation operator E, will be used to construct confidence intervals for the parameters, see Cohen [18]. The observed Fisher information matrix has second partial derivatives of log-likelihood function as the entries, which easily can be obtained. Hence, the observed information matrix is given by;

$$\hat{I}(\alpha,\gamma,\theta)=\begin{pmatrix}-\frac{\partial^2\ell(\alpha,\gamma,\theta\mid \underline{y})}{\partial\alpha^2} & -\frac{\partial^2\ell(\alpha,\gamma,\theta\mid \underline{y})}{\partial\alpha\partial\gamma} & -\frac{\partial^2\ell(\alpha,\gamma,\theta\mid \underline{y})}{\partial\alpha\partial\theta}\\ -\frac{\partial^2\ell(\alpha,\gamma,\theta\mid \underline{y})}{\partial\gamma\partial\alpha} & -\frac{\partial^2\ell(\alpha,\gamma,\theta\mid \underline{y})}{\partial\gamma^2} & -\frac{\partial^2\ell(\alpha,\gamma,\theta\mid \underline{y})}{\partial\gamma\partial\theta}\\ -\frac{\partial^2\ell(\alpha,\gamma,\theta\mid \underline{y})}{\partial\theta\partial\alpha} & -\frac{\partial^2\ell(\alpha,\gamma,\theta\mid \underline{y})}{\partial\theta\partial\gamma} & -\frac{\partial^2\ell(\alpha,\gamma,\theta\mid \underline{y})}{\partial\theta^2}\end{pmatrix}_{\downarrow(\alpha=\hat{\alpha},\gamma=\hat{\gamma},\theta=\hat{\theta})}. \tag{18}$$

Therefore, the asymptotic variance--covariance matrix for the MLEs is obtained by inverting the observed information matrix $\hat{I}(\alpha,\gamma,\theta)$. Or equivalent.

$$\hat{I}^{-1}(\alpha,\gamma,\theta)=\begin{pmatrix}\widehat{\operatorname{var}(\alpha)} & \operatorname{cov}(\alpha,\gamma) & \operatorname{cov}(\alpha,\theta)\\ \operatorname{cov}(\gamma,\alpha) & \widehat{\operatorname{var}(\gamma)} & \operatorname{cov}(\gamma,\theta)\\ \operatorname{cov}(\theta,\alpha) & \operatorname{cov}(\theta,\gamma) & \widehat{\operatorname{var}(\theta)}\end{pmatrix}_{\downarrow(\hat{\alpha},\hat{\gamma},\hat{\theta})}. \tag{19}$$

It is well known that under some regularity conditions, see Lawless [19], $(\hat{\alpha},\hat{\gamma},\hat{\theta})$ is approximately distributed as multivariate normal with mean (α, γ, θ) and covariance matrix $I^{-1}(\alpha,\gamma,\theta)$. Thus, the $(1-\delta)100\%$ approximate confidence intervals (ACIs) for α, γ and θ can be given by;

$$\left(\hat{\alpha}\pm Z_{\delta/2}\sqrt{\widehat{\operatorname{var}(\alpha)}}\right),\quad \left(\hat{\gamma}\pm Z_{\delta/2}\sqrt{\widehat{\operatorname{var}(\gamma)}}\right),\quad \left(\hat{\theta}\pm Z_{\delta/2}\sqrt{\widehat{\operatorname{var}(\theta)}}\right) \tag{20}$$

where $Z_{\delta/2}$ is the percentile of the standard normal distribution with right-tail probability $\delta/2$. The problem with applying normal approximation of the MLE is that when the sample size is small, the normal approximation may be poor. However, a different transformation of the MLE can be used to correct the inadequate performance of the normal approximation. Meeker and Escobar [20] suggested the use of the normal approximation for the log-transformed MLE. Thus, A two-sided $(1-\delta)100\%$ normal approximation CIs for $\Omega = (\alpha,\gamma,\theta)$ are given by;

$$\left(\hat{\Omega}.\exp\left\{-\frac{Z_{\frac{\delta}{2}}\sqrt{\widehat{\operatorname{var}(\hat{\Omega})}}}{\hat{\Omega}}\right\},\quad \hat{\Omega}.\exp\left\{\frac{Z_{\frac{\delta}{2}}\sqrt{\widehat{\operatorname{var}(\hat{\Omega})}}}{\hat{\Omega}}\right\}\right), \tag{21}$$

where $\hat{\Omega}=(\hat{\alpha},\hat{\gamma},\hat{\theta})$.

Parametric Bootstrap Methods: The bootstrap is a resampling method for statistical inference. It is commonly used to estimate confidence intervals, but it can also be used to estimate bias and variance of an estimator or calibrate hypothesis tests. Also it is evident that the confidence intervals based on the asymptotic results do not perform very well for small sample size. For this, we propose to use confidence intervals based on the parametric bootstrap methods. We present two parametric bootstrap methods, (i) percentile bootstrap method (we call it PB) based on the idea of Efron [21]. (ii) bootstrap-t method (we call it BT) based on the idea of Hall [22]. For more survey of the parametric bootstrap methods, see Davison and Hinkley [23] and a more recently reviewed article by Kreiss and Paparoditis.[24]. The following steps are followed to obtain bootstrap samples for both methods:

- Based on the original progressively type-II sample, $\underline{y}\equiv y^{R}_{1;m,n}<\ldots<y^{R}_{n_1;m,n}<y^{R}_{n_1+1;m,n}<\ldots<y^{R}_{m;m,n}$ compute $\hat{\alpha},\hat{\gamma}$ and $\hat{\theta}$.
- Use $\hat{\alpha},\hat{\gamma}$ and $\hat{\theta}$ to generate a bootstrap sample $\underline{y}^*$ with the same values of R_i, $i = 1,2,..,m$ using algorithm presented in Balakrishnan and Sandhu [25].
- As in Step 1 based on $\underline{y}^*$ compute the bootstrap sample estimates of $\hat{\alpha},\hat{\gamma}$ and $\hat{\theta}$ say $\hat{\alpha}^*,\hat{\gamma}^*$ and $\hat{\theta}^*$.
- Repeat the above Steps 2 and 3 N times and arrange all $\hat{\alpha}^*,\hat{\gamma}^*$ and $\hat{\theta}^*$ in ascending order to obtain the bootstrap sample $(\widehat{\Psi}_k^{*[1]},\ \widehat{\Psi}_k^{*[2]},\ldots,\ \widehat{\Psi}_k^{*[N]}),k=1,2,3$, where $\widehat{\Psi}_1^*=\hat{\alpha}^*,\widehat{\Psi}_2^*=\hat{\gamma}^*$ and $\widehat{\Psi}_3^*=\hat{\theta}^*$.

Percentile Bootstrap Procedure: Let $\Phi(z)=P(\widehat{\Psi}_k^*\le z)$ be the cumulative distribution function of $\widehat{\Psi}_k^*$. Define $\widehat{\Psi}_{kBoot}^*=\Phi^{-1}(z)$ for given z The approximate bootstrap $100(1-\delta)100\%$ confidence interval of $\widehat{\Psi}_k^*$ is given by;

$$\left[\widehat{\Psi}^{*}_{kPB}\left(\frac{\delta}{2}\right),\widehat{\Psi}^{*}_{kPB}\left(1-\frac{\delta}{2}\right)\right]. \tag{22}$$

Bootstrap-T Procedure: We find the order statistics $\mu_k^{*[1]} < \mu_k^{*[2]} < ... < \mu_k^{*[N]}$ where,

$$\mu_k^{*[j]} = \frac{\sqrt{N}\left(\widehat{\Psi}_k^{*[j]} - \widehat{\Psi}_k\right)}{\sqrt{Var(\widehat{\Psi}_k^{*[j]})}}, \ j = 1,2,...,N, \ k = 1,2,3, \tag{23}$$

where $\widehat{\Psi}_1 = \hat{\alpha}, \widehat{\Psi}_2 = \hat{\gamma}$, $\widehat{\Psi}_3 = \hat{\theta}$ and $Var(\widehat{\Psi}_k^{*[j]})$ is obtained using the Fisher information matrix. Let $W(z) = P(\mu_k^{*} < z), k = 1,2,3$ be the cumulative distribution function of μ_k^{*}. For a given z, define

$$\widehat{\Psi}_{kBT} = \widehat{\Psi}_k + N^{-1/2}\sqrt{Var(\widehat{\Psi}_k)}W^{-1}(z). \tag{24}$$

Thus, The approximate bootstrap 100(1–δ)% confidence interval of $\widehat{\Psi}_k$ is given by

$$\left[\widehat{\Psi}^{*}_{kBT}\left(\frac{\delta}{2}\right),\widehat{\Psi}^{*}_{kBT}\left(1-\frac{\delta}{2}\right)\right]. \tag{25}$$

Bayes Estimation: In this section we obtain Bayesian estimates and the corresponding credible intervals of the unknown parameters α, γ and θ. Let us consider independent vague priors for the parameters α, γ and θ, as follows:

$$\begin{cases} \pi(\alpha) \propto \alpha^{-1}, & \alpha > 0, \\ \pi(\gamma) \propto \gamma^{-1} & \gamma > 0, . \\ \pi(\theta) \propto \theta^{-1}, & \theta > 1. \end{cases} \tag{26}$$

Therefore, the joint prior of the parameters α , γ and θ can be expressed by;

$$\pi(\alpha,\gamma,\lambda) \propto (\alpha\gamma\theta)^{-1}, \qquad \alpha > 0, \gamma > 0, \theta > 1. \tag{27}$$

It is to be noted that our objective is to consider vague priors so that the priors do not have any significant roles in the analysis that follow. However, if one uses the prior beliefs different from (26) and resorts to sample based approaches for analyzing the posterior, one may use the concept of sampling-importance-resampling without working afresh with the new prior-likelihood setup, see Upadhyay *et al*. [26]. The joint posterior density function of the parameters α, γ and θ denoted by $\pi^{*}(\alpha, \beta, \lambda \mid x)$, up to proportionality can be obtained by combining the likelihood function (10) with the prior (27) via Bayes' theorem and it can be written as:

$$\pi^{*}(\alpha,\gamma,\theta \mid \underline{y}) = \frac{L(\alpha,\gamma,\theta \mid \underline{y}) \times \pi(\alpha,\gamma,\theta)}{\int_0^\infty\int_0^\infty\int_0^\infty L(\alpha,\gamma,\theta \mid \underline{y}) \times \pi(\alpha,\beta,\lambda) d\alpha d\gamma d\theta}. \tag{28}$$

Therefore, the Bayes estimate of any function of the parameters, say $g(\alpha, \gamma, \theta)$, under squared error loss function (SELF) can be obtained as:

$$\hat{g}(\alpha,\beta,\lambda)=E_{\alpha,\gamma,\theta|\underline{y}}\left[g(\alpha,\gamma,\theta)\right] \\ =\frac{\int_0^\infty\int_0^\infty\int_0^\infty g(\alpha,\gamma,\theta)\times L(\alpha,\gamma,\theta\mid\underline{y})\times\pi(\alpha,\gamma,\theta)d\alpha d\gamma d\theta}{\int_0^\infty\int_0^\infty\int_0^\infty L(\alpha,\gamma,\theta\mid\underline{y})\times\pi(\alpha,\gamma,\theta)d\alpha d\gamma d\theta}. \tag{29}$$

It may be noted that, the calculation of the multiple integrals in (29) cannot be solved analytically. In this case, we use the MCMC approximation method to generate samples from the joint posterior density function in (28) and then compute the Bayes estimators of the unknown parameters and construct the corresponding credible intervals. To implement the MCMC methodology, we consider the Gibbs within Metropolis sampler, which requires the derivation of the complete set of conditional posterior distribution see Metropolis *et al*. [27] and Tierney [28]. From (28), the joint posterior up to proportionality can be written as:

$$\pi^*(\alpha,\gamma,\theta\mid\underline{y})\propto\alpha^{m-1}\ \gamma^{-1}\ \theta^{(m-n_1-1)}\ \exp\left\{-\frac{\alpha}{\gamma}\left[\sum_{i=1}^{n_1}(R_i+1)y_i^{\gamma}+\sum_{i=n_1+1}^{m}(R_i+1)(\phi_i(\theta))^{\gamma}\right]\right\} \\ \times\left[\prod_{i=1}^{n_1}y_i^{(\gamma-1)}\right]\left[\prod_{i=n_1+1}^{m}(\phi_i(\theta))^{(\gamma-1)}\right], \tag{30}$$

where $\phi_i(\theta)=\tau+\theta(y_i-\tau)$. The full conditionals for α, γ and θ can be written, up to proportionality, as;

$$\pi_1^*(\alpha\mid\gamma,\theta,\underline{y})\equiv\text{Gamma}\left(m,\frac{1}{\gamma}\left[\sum_{i=1}^{n_1}(R_i+1)y_i^{\gamma}+\sum_{i=n_1+1}^{m}(R_i+1)(\phi_i(\theta))^{\gamma}\right]\right), \tag{31}$$

$$\pi_2^*(\gamma\mid\alpha,\theta,\underline{y})\propto\gamma^{-1}\left[\prod_{i=1}^{n_1}y_i^{(\gamma-1)}\right]\left[\prod_{i=n_1+1}^{m}(\phi_i(\theta))^{(\gamma-1)}\right] \\ \times\exp\left\{-\frac{\alpha}{\gamma}\left(\sum_{i=1}^{n_1}(R_i+1)y_i^{\gamma}+\sum_{i=n_1+1}^{m}(R_i+1)(\phi_i(\theta))^{\gamma}\right)\right\}, \tag{32}$$

and

$$\pi_3^*(\theta\mid\alpha,\gamma,\underline{y})\propto\theta^{(m-n_1-1)}\left(\prod_{i=n_1+1}^{m}(\phi_i(\theta))^{(\gamma-1)}\right)\exp\left\{-\frac{\alpha}{\gamma}\left(\sum_{i=n_1+1}^{m}(R_i+1)(\phi_i(\theta))^{\gamma}\right)\right\}. \tag{33}$$

It can be easily seen that the conditional posterior densities of α given in (31) is gamma density with parameters $\left(m,\frac{1}{\gamma}\left[\sum_{i=1}^{n_1}(R_i+1)y_i^{\gamma}+\sum_{i=n_1+1}^{m}(R_i+1)(\phi_i(\theta))^{\gamma}\right]\right)$. Thus, samples of α can be easily generated using any gamma generating routine. Also, since the conditional posteriors of γ and θ in (32) and (33) do not present standard forms, but the plot of both them shows that they similar to normal distribution see Figures 1 and 2, and so Gibbs sampling is not a straightforward option, the use of the Metropolis-Hasting (M-H) sampler is required for the implementations of MCMC methodology. Given these conditional distributions in (31)-(33), below is a hybrid algorithm with Gibbs sampling steps for updating the parameter α and with M-H steps for updating γ and θ. To run the Gibbs sampler algorithm we started with the MLEs of $\hat{\alpha},\hat{\gamma}$ and $\hat{\theta}$. We then drew samples from various full conditionals, in turn, using the most recent values of all other conditioning variables unless some systematic pattern of convergence was achieved. Now, the following steps illustrate the process of the M-H algorithm within Gibbs sampling:

- Start with initial guess $\left(\alpha^{(0)},\gamma^{(0)},\theta^{(0)}\right)$.
- Set j = 1.

Generate $\alpha^{(j)}$ from Gamma $\left(m,\frac{1}{\gamma}\left[\sum_{i=1}^{n_1}(R_i+1)y_i^{\gamma}+\sum_{i=n_1+1}^{m}(R_i+1)\left(\phi_i(\theta)\right)^{\gamma}\right]\right)$

Using the following M-H algorithm, generate $\gamma^{(j)}$ and $\theta^{(j)}$ from $\pi_2^*(\gamma^{(j-1)}\mid\alpha^{(j)},\theta^{(j-1)},\underline{y})$ and $\pi_3^*(\theta^{(j-1)}\mid\alpha^{(j)},\gamma^{(j)},\underline{y})$ with the normal proposal distributions $N\left(\gamma^{(j-1)},\mathrm{var}(\gamma)\right)$ and $N\left(\theta^{(j-1)},\mathrm{var}(\theta)\right)$.

- Generate a proposal γ^* from $N\left(\gamma^{(j-1)},\mathrm{var}(\gamma)\right)$ and θ^* from $N\left(\theta^{(j-1)},\mathrm{var}(\theta)\right)$
- Evaluate the acceptance probabilities

$$\zeta_\gamma=\min\left[1,\frac{\pi_2^*(\gamma^*\mid\underline{y},\alpha^{(j)},\theta^{(j-1)})}{\pi_2^*(\gamma^{(j-1)}\mid\underline{y},\alpha^{(j)},\theta^{(j-1)})}\right],$$
$$\zeta_\theta=\min\left[1,\frac{\pi_3^*(\theta^*\mid\underline{y},\alpha^{(j)},\gamma^{(j)})}{\pi_3^*(\theta^{(j-1)}\mid\underline{y},\alpha^{(j)},\gamma^{(j)})}\right]. \tag{34}$$

- Generate a u_1 and u_2 from a Uniform (0, 1) distribution
- If $u_1<\zeta_\gamma$ accept the proposal and set $\gamma^{(j)}=\gamma^*$, else set $\gamma^{(j)}=\gamma^{(j-1)}$
- If $u_1<\zeta_\theta$ accept the proposal and set $\theta^{(j)}=\theta^*$, else set $\theta^{(j)}=\theta^{(j-1)}$
- Set $j=j+1$
- Repeat Steps (3)-(5) N times.

In order to guarantee the convergence and to remove the affection of the selection of initial value, the first M simulated varieties are discarded. Then the selected sample are $\alpha^{(j)}$, $\gamma^{(j)}$ and $\theta^{(j)}$, $j=M+11,,...,N$, for sufficiently large N, forms an approximate posterior sample which can be used to develop the Bayes estimates of $\xi=\alpha$, γ or θ as;

$$\hat{\xi}_{MC}=\frac{1}{N-M}\sum_{j=M+1}^{N}\xi^{(j)}. \tag{35}$$

To compute the credible intervals of α, γ and θ, order $\alpha^{(i)}$, $\gamma^{(i)}$ and $\theta^{(i)}$ i=1,...,N as $\left\{\alpha^{(1)}<\ldots<\alpha^{(N)}\right\}$, $\left\{\gamma^{(1)}<\ldots<\gamma^{(N)}\right\}$, and $\left\{\theta^{(1)}<\ldots<\theta^{(N)}\right\}$. Then the 100(1−$\delta$)% credible intervals of $\xi=\alpha$, γ or θ become.

$$[\xi_{(N(\delta/2))},\xi_{(N(1-\delta/2))}]. \tag{36}$$

RESULTS AND DISCUSSION

Numerical Computations: In this section, we present a simulation example to check the estimation procedures. In this example, by using the algorithm described in Balakrishnan and Sandhu [23], we generate a progressive type-II censored sample from DPHF(α, γ) under SSPALT model. The generation of a progressive type-II censored sample from DPHF under SSPALT model is performed according to the following algorithm.

- Specify the values of n, m and R_i, i=1,2,...m.
- Specify the values of the parameters α, γ and θ.

- Specify the values of the optimal stress change time τ
- Generate a random sample with size n and censoring size m from the random variable Y given by (6), the set of data can be considered as:

$$y_{1;m,n}^{R} < \dots < y_{n_1;m,n}^{R} < y_{n_1+1;m,n}^{R} < \dots < y_{m;m,n}^{R},$$

where $R = (R_1, R_2, \dots, R_m)$ and $\sum_{i=1}^{m} R_i = n - m.$

- Use the progressive type-II censored sample to compute the MLEs of the model parameters and the acceleration factor. The Newton-Raphson method is applied for solving the nonlinear system to obtain the MLEs of the parameters α, γ and θ.
- Compute the 95% bootstrap confidence intervals for the model parameters and the acceleration factor, using the steps described in Section 4.
- Compute the Baye estimates of the parameters α, γ and θ based on MCMC algorithm described in Section 5.

A simulation data for progressive type-II censored sample under SSPALT model from DPHF with true values α = 0.5, γ = 1.5, acceleration factor θ = 2 and τ = 0.90, using progressive censoring schemes n = 40, m = 24 and R= (2, 0, 1, 2, 0, 0, 1, 0, 0, 1, 0, 1, 2, 0, 0, 1, 0, 2, 0, 1, 0, 2, 0, 0) has been truncated after four decimal places and it has been presented in Table 1. In the MCMC approach, we run the chain for 32000 times and discard the first 2000 values as `burn-in'. Figures 1 and 2 plots the posterior density functions $\pi_2^*(\gamma \mid \alpha, \theta, \underline{y})$ and $\pi_3^*(\theta \mid \alpha, \gamma, \underline{y})$. The MLEs $(.)_{ML}$, bootstrap $(.)_{PB}$, $(.)_{BT}$ and Bayes MCMC $(.)_{MCMC}$ point estimates of the parameters are obtained and presented in Table 2. The approximate confidence intervals (ACIs), bootstrap confidence intervals (PBCIs, BTCIs) and credible intervals (CRIs) for the parameters α, γ and θ are computed. The results of 95% ACIs, PBCIs, BTCIs, and CRIs are presented in Table 3. Figures 3-5 show simulation number of α, γ and θ generated by MCMC samples (dashed lines (...) represent the posterior means and soled lines (-) represent lower and upper bounds 95% probability interval.) and the corresponding histograms in Figures 6-8. A sample of size 30 000 is the obtained to make approximate Bayesian inference including posterior mean, median, mode and credible interval of the parameters of interest constructed by the 2.5% and 97.5% quantities. The MCMC results of the posterior mean, median, mode, standard deviation (S.D) and skewness (Ske) of the parameters α, γ and θ are displayed in Table 4.

Table 1: SSPALT simulation data with true values for α γ, θ and τ

Under normal condition			Under accelerated condition				
0.1579	0.2368	0.2557	0.9297	0.9319	1.1055	1.1584	1.1823
0.4285	0.4866	0.5579	1.2026	1.2510	1.2676	1.2714	1.2759
0.6699	0.6714	0.8488	1.5194	1.5976	1.5980	2.1318	3.1422

Table 2: Different point estimates for (α, γ, θ) = (0.5,1.5,2)

Parameters	$(.)_{ML}$	$(.)_{PB}$	$(.)_{BT}$	$(.)_{MCMC}$
α	0.5077	0.5261	0.5144	0.4972
γ	1.4755	1.5513	1.5222	1.4666
θ	1.9497	2.0839	1.9797	1.9628

Table 3: 95% confidence intervals for α, γ and θ

Method	α	Length	γ	Length	θ	Length
ACI	[0.1945, 1.3247]	1.1302	[0.8802, 2.4735]	1.5933	[0.6026, 6.3078]	5.7052
PBCI	[0.2354, 1.4482]	1.2128	[0.9233, 2.7398]	1.8165	[0.7858, 6.5213]	5.7355
BTCI	[0.1388, 1.0982]	0.9594	[0.5764, 2.1155]	1.5391	[0.8427, 4.1206]	3.2779
CRI	[0.3157, 0.7140]	0.3983	[1.0473, 1.9205]	0.8732	[0.9223, 3.4492]	2.5269

Table 4: MCMC results for α, γ and θ

Parameters	Mean	Median	Mode	Variance	S.D	Ske
α	0.4972	0.4891	0.4759	0.0110	0.1050	0.4403
γ	1.4666	1.4601	1.4477	0.0495	0.2226	0.2341
θ	1.9628	1.9588	1.9357	0.0521	0.2282	0.3547

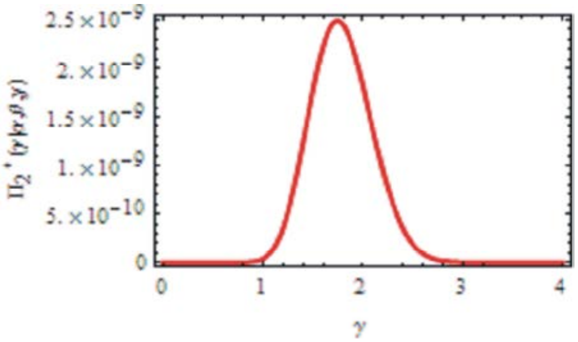

Fig. 1: Posterior density function $\pi_2^*(\gamma|\alpha,\theta,\underline{y})$ of γ.

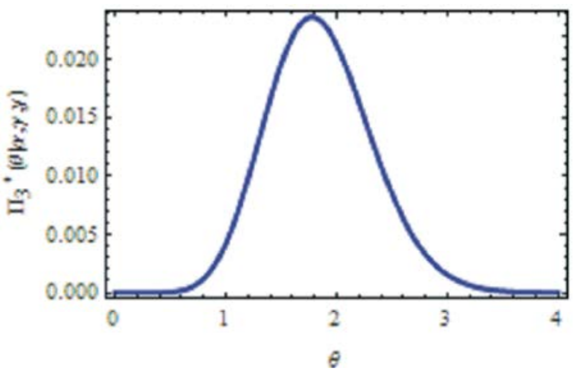

Fig. 2: Posterior density function $\pi_3^*(\gamma|\alpha,\theta,\underline{y})$ of θ

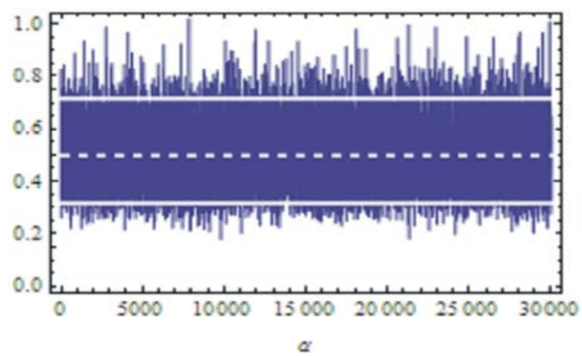

Fig. 3: Simulation number of α gene-rated by the MCMC method

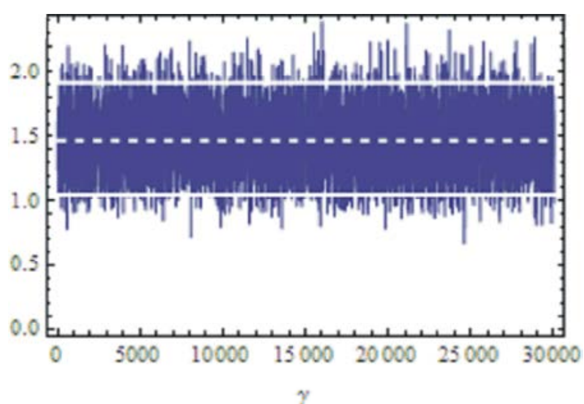

Fig. 4: Simulation number of γ gene-rated by the MCMC method

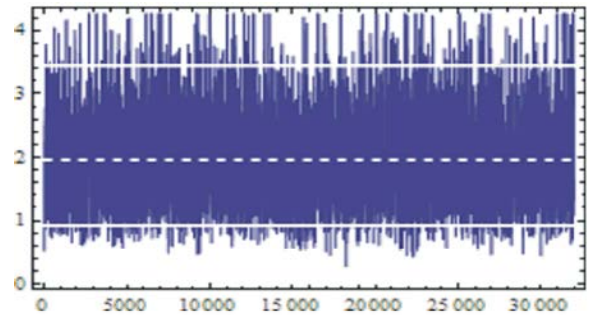

Fig. 5: Simulation number of θ gene-rated by the MCMC method

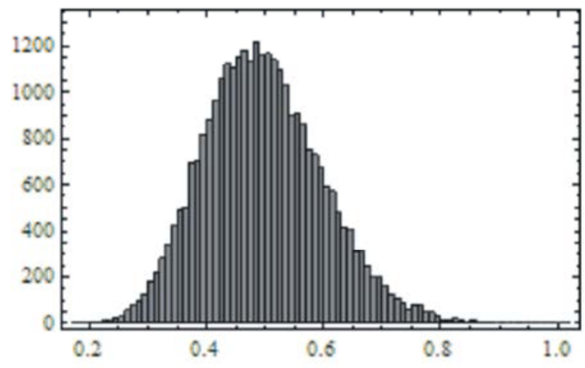

Fig. 6: Histogram of α generated by the MCMC method

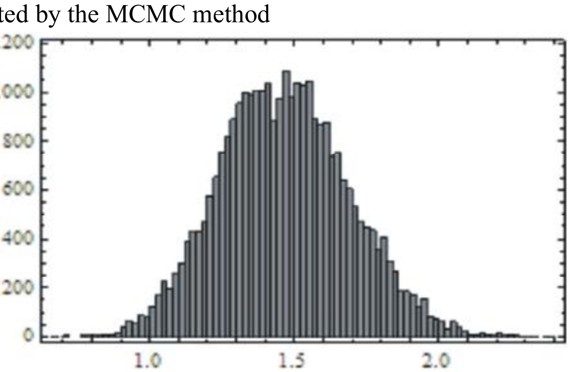

Fig. 7: Histogram of γ generated by the MCMC method

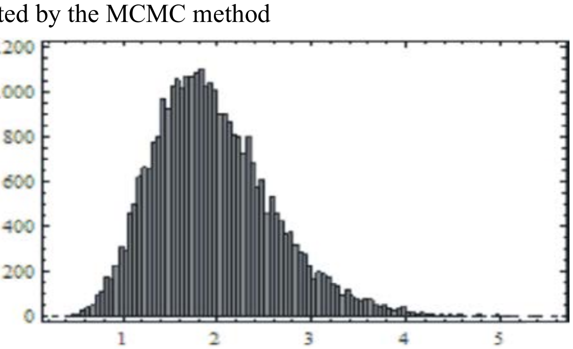

Fig. 8: Histogram of θ generated by the MCMC method

CONCLUSION

Based on progressively type-II censored samples, this article is a related to full Bayes and non-Bayes procedure for the analysis of the SSPALT using the DPHF failure model. The classical Bayes estimates cannot be obtained in explicit form. One can clearly see the scope of MCMC-based Bayesian solutions which make every inferential development routinely available. In this article, we have considered the maximum likelihood and Bayes estimates for the parameters of DPHF using progressive type-II censored scheme. This article also studied the construction of confidence intervals for the parameters by using the parametric bootstrap methods. It well known that when all parameters are unknown, the Bayes estimates cannot be obtained in explicit form. The MCMC and parametric bootstrap techniques are used to compute the approximate Bayes estimates and the corresponding credible intervals. A numerical example using the simulated data set is presented to illustrate how the MCMC and parametric bootstrap methods are work based on progressive censored data under SSPALT model.

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