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# Deflection of Free Opposite Edges Rectangular Plates Using Minimum Energy Concept

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**Abstract:** This paper introduces new displacement functions for rectangular plate in general coordinates under the entire applicable boundary conditions with any in plane or out-plane loading. These developed functions are easy and fast to be used by engineers in construction sites or manufacturing facilities by just simple calculator. The method of analysis depends mainly on the minimum energy concept and appropriate real polynomials in the functions at the points coordinates over the plate area. The current study introduces then a simple method of analysis seeking acceptable and accurate results.

**Key words:** Plate · Deformation · Free edge · Minimum energy concept

## INTRODUCTION

The first drive to a mathematical expression of the plate problems, was done by Euler [1], where he carried on a free vibration analysis of the plate problems. From then several trials and developments were carried on to solve plate problems. The German physicist Chladni [2] found out the free vibrations various modes. Bernoulli's solution [3] was depending on the previous work, which results in the Euler-Bernoulli's bending beam theory. Bernoulli showed a plate as a system of interactive perpendicular strips to one another, each strip was considered as a beam function. However, the governing differential function, as found from this procedure, did not contain the middle term (described in details at [3]). Germain, the French mathematician (1809) improved a plate differential function that miss the warping term (described in details at [4]). Lagrange [5] corrected her results by adding the lacked term; therefore, he was the first to develop the "*general plate equation*" correctly. was the first to describe applicable theory of bending of plates. He considered, in the general plate equation, the thickness *t* of plate as a rigidity function. Moreover, he also presented an "*exact method*'', which turned the differential equation by use of "*Fourier trigonometric series*" into algebraic expressions. The first to set the formula for the problem of plate bending were Cauchy [7] and Poisson [8], that was based on theory of elasticity general equations. All characteristic quantities were expanded into series in powers of how far from the midline surface. They keep only the first order small terms. In this manner, they achieved the governing differential equation for deflections that concur greatly with the "*Germain–Lagrange plate equation*". Poisson [8] was successfully able to expand the "*Germain– Lagrange equation*" to a plate solved under static loading case. The plate flexural rigidity *D* was set as a constant term in the last solution. In addition, Poisson recommended that any point on a free boundary get up three boundary conditions. These boundary conditions performed by Poisson were the subject of further investigations. Kirchhoff [9] published a thesis on the theory of thin plates in which, he set two independent basic assumptions known as "*Kirchhoff's hypotheses*", that are nowadays greatly accepted in the plate-bending theory. Kirchhoff simplified the energy functional of "*3D elasticity theory*" for deformed plates by the use of these hypotheses. Who obtained the "*Germain-Lagrange equation*" as the Euler equation by requiring that it should be stationary. In addition, he showed that there exist only two boundary conditions on any plate edge. Moreover, his discovery of "the frequency equation" of plates and "virtual displacement

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methods" helped in the solution of plate problems. Kelvin and Tait [10] presented an extra discernment based on the boundary condition equations by converting twisting moments along the plate edges into shearing forces. Therefore, each edge is subjected to only two forces (shear and moment). Of course, a consequential contribution was made by Timoshenko and Woinowsky-Krieger [11] to the application and theory of plate bending analysis. Levy [12], in the 19th century, was succeed in solving rectangular plates problem with two parallel simple supported edges and the other two supports arbitrary.

## MATERIALS AND METHODS

The present study of the rectangular plate will depend mostly on the separation of variables in setting the new deflection *w* expression. Therefore, the deflection expression *w* will be formed as multiplication of two functions *f* and *g*. Each of the two functions will be a function in only one of the two different variables *x* and *y* presenting the plate coordinates separately (i.e. $w = f(x)$. The separation will make it easier to apply the boundary conditions separately on the two functions of each edge (side) without affecting the other one besides simplifying the calculation when both functions are needed together. The used assumed function of the deflection *w* will be set as two separate polynomial functions instead of tough and complicated expressions like trigonometric and logarithmic...etc. Therefore, in the present case study;

Function $f(x)$ will be set as:

$$f(x) = c_5x^4 + c_4x^3 + c_3x^2 + c_2x + c_1 \quad (1)$$

and function $g(y)$ will be set as:

$$g(y) = c_{10}y^4 + c_9y^3 + c_8y^2 + c_7y + c_6 \quad (2)$$

and $c_i$ (where $i$ = 1 ,2,3,...,10) are constants to be set in such manner satisfying the boundary conditions of the plate at all sides (the number of these constants can be increased if necessary). Besides the constants are all solved in such a way that they extermize the total potential energy function of the system marked as $\pi$ for rectangular plate [13]. The resulting assumed function of *w* given by the following Eq. (3) will consist of twenty-five terms:

$$w = (c_5x^4 + c_4x^3 + c_3x^2 + c_2x + c_1)(c_{10}y^4 + c_9y^3 + c_8y^2 + c_7y + c_6) \quad (3)$$

which gives an acceptable predicted accuracy. The present results will be compared with symplectic method [14].

**Case (1): (SSFF) Rectangular Plate $a \times b$ :** Considering the plate bounded by the boundaries of domain ($0 \le x \le a$), which are two opposite simply supported edges and the boundaries of domain ($0 \le y \le b$), which are two opposite free edges, as seen in Fig. 1. First, it is needed to get the constants values of the assumed function $f(x)$ such that they satisfy the boundary conditions of SS edges. Therefore,

$$f(0) = f(a) = \left.\frac{d^2 f(x)}{dx^2}\right|_{x=0} = \left.\frac{d^2 f(x)}{dx^2}\right|_{x=a} = 0 \quad (4)$$

The results of equations Eq. (4) are solved to get the constants. Therefore,

$$w = c_5(a^3x - 2ax^3 + x^4)\ (c_{10}y^4 + c_9y^3 + c_8y^2 + c_7y + c_6) \quad (5)$$

the bending moment $M_y$ and the effective shear force $V_y$ are;

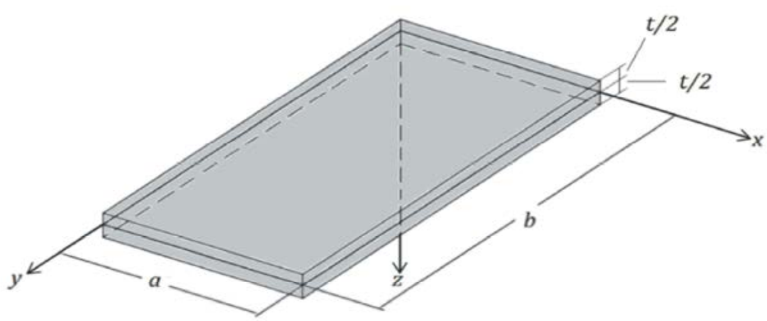


Fig. 1: The coordinate and dimensions of the rectangular plate $a \times b$ used in the study

$$M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + v\left(\frac{\partial^2 w}{\partial x^2}\right)\right) = -Dc_5\big((a^3x - 2ax^3 + x^4)(12y^2c_{10} + 6yc_9 + 2c_8) + v(-12ax + 12x^2)(y^4c_{10} + y^3c_9 + y^2c_8 + yc_7 + c_6)\big) \quad (6)$$

$$V_y = -D\left(\frac{\partial^3 w}{\partial y^3} + (2-v)\left(\frac{\partial^3 w}{\partial x^2 \partial y}\right)\right) = -Dc_5\big((a^3x - 2ax^3 + x^4)(24yc_{10} + 6c_9) + (2-v)(-12ax + 12x^2)(4y^3c_{10} + 3y^2c_9 + 2yc_8 + c_7)\big) \quad (7)$$

where $v$ is the Poisson's ratio, $D = \frac{Et^3}{12(1-v^2)}$ is the modulus of rigidity, $E$ Young's modulus and t is the plate thickness.

Moreover, the boundary conditions for the free edges require;

$$M_y\left(x = \tfrac{1}{2}a, y = 0\right) = 0 \rightarrow \left(\tfrac{5}{8}\right)a^4c_8 - 3va^2c_6 = 0, \quad (8)$$

$$M_y\left(x = \tfrac{1}{2}a, y = b\right) = 0 \rightarrow \left(\tfrac{5}{16}\right)a^4(12b^2c_{10} + 6bc_9 + 2c_8) - 3va^2(b^4c_{10} + b^3c_9 + b^2c_8 + bc_7 + c_6) = 0 \quad (9)$$

$$V_y\left(x = \tfrac{1}{2}a, y = 0\right) = 0 \rightarrow \left(\tfrac{15}{8}\right)a^4c_9 - 3(2-v)a^2c_7 = 0, \quad (10)$$

and

$$V_y\left(x = \tfrac{1}{2}a, y = b\right) = 0 \rightarrow \left(\tfrac{5}{16}\right)a^4(24bc_{10} + 6c_9) - 3(2-v)a^2(4b^3c_{10} + 3b^2c_9 + 2bc_8 + c_7) = 0 \quad (11)$$

The results of Eq. (8) to Eq. (11) are solved to get the constants as;

$$c_6 = c_{10}\left(\frac{-5}{96}\right)a^2\left(\frac{-4b^2v + 5a^2 + 8b^2}{v(v-2)}\right), c_7 = c_{10}\left(\frac{5}{4}\right)\left(\frac{a^2b}{v-2}\right), c_8 = c_{10}\left(\frac{-1}{4}\right)\left(\frac{-4b^2v + 5a^2 + 8b^2}{v-2}\right) \text{ and } c_9 = -2bc_{10} \quad (12)$$

Substituting the above obtained constants in Eq. (2) yields

$$g(y) = c_{10}\left(y^4 - 2y^3b + \left(\frac{1}{4}\right)y^2\left(\frac{4b^2v - 5a^2 - 8b^2}{v-2}\right) + \left(\frac{5a^2}{4}\right)y\left(\frac{b}{v-2}\right) + \left(\frac{5a^2}{96}\right)\left(\frac{4b^2v - 5a^2 - 8b^2}{v(v-2)}\right)\right) \quad (13)$$

Therefore, Eq. (3) becomes

$$w = c(a^3x - 2ax^3 + x^4)\left(y^4 - 2y^3b + \left(\frac{1}{4}\right)y^2\left(\frac{4b^2v - 5a^2 - 8b^2}{v-2}\right) + \left(\frac{5}{4}\right)ya^2\left(\frac{b}{v-2}\right) + \left(\frac{5}{96}\right)a^2\left(\frac{4b^2v - 5a^2 - 8b^2}{v(v-2)}\right)\right)$$

in which $c = c_5 \times c_{10}$ (14)

Consider the case of plate under uniformly distributed load $q$ over whole area. Therefore, the total potential energy $\pi$ [13] is given by:

$$\pi = \left(\frac{D}{2}\right)\iint_{0\,0}^{a\,b}\left\{\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)^2+2(1-v)\left(\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2-\left(\frac{\partial^2 w}{\partial y^2}\right)\left(\frac{\partial^2 w}{\partial y^2}\right)\right)\right\}dy\,dx-\iint_{0\,0}^{a\,b} qw\,dy\,dx$$

Therefore,

$$\pi = \frac{-25}{7056}\left(\frac{cba^5}{v^2(v-2)^2}\right)(-5.55Da^4b^4cv^4-2.61Da^2b^6cv^4-1.08Db^8cv^4-71.4Da^6b^2cv^3-0.63Da^4b^4cv^3-11.06Da^2b^6cv^3+4.38Db^8cv^3+45.85Da^8cv^2+7.14Da^6b^2cv^2-88.24Da^4b^4cv^2+513792Da^2b^6cv^2-4.3Db^8cv^2+147Da^6b^2cv+211.7Da^4b^4cv-37.63Da^2b^6cv-45.94Da^8c-147Da^6b^2c-117.6Da^4b^4c+1.88b^4v^4q+23.52a^2b^2v^3q-7.53b^4v^3q-14.7a^4v^2q-70.56a^2b^2v^2q+7.53b^4v^2q+29.4a^4vq+47a^2b^2vq) \quad (15)$$

$\pi$ is extremum (minimum) when $\frac{\partial \pi}{\partial t}=0$, which gives:

$$c = 588qv\mu(16b^4v^3+200a^2b^2v^2-64b^4v^2-125a^4v-600a^2b^2v+64b^4v+250a^4+400a^2b^2)D^{-1} \quad (16)$$

as:

$$\mu = \left((0.56a^4b^4v^4+0.26a^2b^6v^4+0.11b^8v^4+7.14a^6b^2v^3+0.063a^4b^4v^3+1.11a^2b^6v^3-0.43b^8v^3-4.59a^8v^2-7.14a^6b^2v^2+8.82a^4b^4v^2-5.14a^2b^6v^2+0.43b^8v^2-14.7a^6b^2v-21.17a^4b^4v+3.76a^2b^6v+4.59a^8+14.7a^6b^2+11.76a^4b^4)\times 10^5\right)^{-1} \quad (17)$$

Substituting the constant $c$ obtained in Eq. (14) gives the final deflection expression $w$ as:

$$w = -\left(\frac{49}{8D}\right)q\mu(-96b^2v^2y^2+192bv^2y^3-96v^2y^4-20a^2b^2v-120a^2bvy+120a^2vy^2+192b^2vy^2-384bvy^3+192vy^4+25a^4+40a^2b^2)(a^3-2ax^2+x^3)x(-16b^4v^2-200a^2b^2v+32b^4v+125a^4+200a^2b^2) \quad (18)$$

and the bending moments $M_x$ and $M_y$ as:

$$M_x = \left(\frac{147}{2}\right)(-16a^3b^2v^3+96a^3bv^3y-96a^3v^3y^2+32ab^2v^3x^2-192abv^3x^2y+192av^3x^2y^2-16b^2v^3x^3+96bv^3x^3y-96v^3x^3y^2+20a^5v^2+32a^3b^2v^2-192a^3bv^2y-40a^3v^2x^2+192a^3v^2y^2+20a^2v^2x^3-64ab^2v^2x^2+96ab^2v^2y^2+384abv^2x^2y-192abv^2y^3-384av^2x^2y^2+96av^2y^4+32b^2v^2x^3-96b^2v^2xy^2-192bv^2x^3y+192bv^2xy^3+192v^2x^3y^2-96v^2xy^4+20a^3b^2v+120a^3bvy-120a^3vy^2-20a^2b^2vx-120a^2bvxy+120a^2vxy^2-192ab^2vy^2+384abvy^3-192avy^4+192b^2vxy^2-384bvxy^3+192vxy^4-25a^5+25a^4x-40a^3b^2+40a^2b^2x)x(-16b^4v^2-200a^2b^2v+32b^4v+125a^4+200a^2b^2)q\mu \quad (19)$$

and

$$M_y = -\left(\frac{147}{2}\right)(-96ab^2v^2y^2 + 192abv^2y^3 - 96av^2y^4 + 96b^2v^2xy^2 - 192bv^2xy^3 + 96v^2xy^4 - 4a^3b^2v - 216a^3bvy + 216a^3vy^2 + 20a^2b^2vx + 120a^2bvxy - 120a^2vxy^2 - 32ab^2vx^2 + 192ab^2vy^2 + 192abvx^2y - 384abvy^3 - 192avx^2y^2 + 192avy^4 + 16b^2vx^3 - 192b^2vxy^2 - 96bvx^3y + 384bvxy^3 + 96vx^3y^2 - 192vxy^4 + 5a^5 - 25a^4x + 8a^3b^2 + 192a^3by + 40a^3x^2 - 192a^3y^2 - 40a^2b^2x - 20a^2x^3 + 64ab^2x^2 - 384abx^2y + 384ax^2y^2 - 32b^2x^3 + 192bx^3y - 192x^3y^2)x(-16b^4v^2 - 200a^2b^2v + 32b^4v + 125a^4 + 200a^2b^2)vq\mu \tag{20}$$

Table 1: Deflection and bending moment factors $\alpha$ $\beta$ and $\gamma$ of uniformly loaded SSFF rectangular plate $a \times b$ with $v = 0.3$ for present study and Symplectic method [14] at plate center

| Aspect ratio $\frac{a}{b}$ | $w = a\frac{qb^4}{D}$ | | | $M_x = \beta qb^2$ | | | $M_y = \gamma qb^2$ | | |
|---|---|---|---|---|---|---|---|---|---|
| | at plate *center* ($x = 0.5a$, $y = 0.5b$) | | | | | | | | |
| | $\alpha$ | | | $\beta$ | | | $\gamma$ | | |
| | Present | [14] | $\varepsilon$ | Present | [14] | $\varepsilon$ | Present | [14] | $\varepsilon$ |
| 2/3 | 0.0026500 | 0.0025477 | 4.0 | 0.0579453 | 0.0546 | 6.1 | 0.01952385 | 0.0151 | 19 |
| 1.5 | 0.0682493 | 0.0681020 | 0.2 | 0.2787449 | 0.2769 | 0.7 | 0.04585265 | 0.0407 | 13 |
| 1 | 0.0132372 | 0.0130940 | 1.1 | 0.1253520 | 0.1225 | 2.3 | 0.03237516 | 0.0271 | 19 |
| 2 | 0.2195266 | 0.2194097 | 0.1 | 0.4955701 | 0.4945 | 0.2 | 0.05374628 | 0.0486 | 11 |
| 3 | 1.1333532 | 1.1334448 | 0.0 | 1.1185097 | 1.1186 | 0.0 | 0.06133887 | 0.0552 | 11 |
| 4 | 3.6137824 | 3.6144728 | 0.0 | 1.9924892 | 1.9934 | 0.0 | 0.06454648 | 0.0570 | 13 |
| 5 | 8.8627411 | 8.8646689 | 0.0 | 3.1168416 | 3.1183 | 0.0 | 0.06615132 | 0.0575 | 15 |

Table 2: Deflection and bending moment factors $\alpha$, $\beta$ and $\gamma$ of uniformly loaded SSFF rectangular plate $a \times b$ with $v = 0.3$ for present study and Symplectic method [14] at opposite free edges

| $\frac{a}{b}$ | at opposite *free* edges ($x = 0.5a$, $y = 0$) and ($x = 0.5a$, $y = b$) | | | | | | |
|---|---|---|---|---|---|---|---|
| | $\alpha$ | | | $\beta$ | | | $\gamma$ |
| | Present | [14] | $\varepsilon$ | Present | [14] | $\varepsilon$ | |
| 2/3 | 0.002780343 | 0.00299418 | 7.1 | 0.05465043 | 0.0588431 | 7.1 | 0 |
| 1.5 | 0.074561129 | 0.07489906 | 0.5 | 0.28949601 | 0.2905851 | 0.4 | 0 |
| 1 | 0.014719066 | 0.01501126 | 1.9 | 0.12858577 | 0.1310877 | 1.9 | 0 |
| 2 | 0.233901086 | 0.23431397 | 0.2 | 0.51083997 | 0.5112501 | 0.1 | 0 |
| 3 | 1.172559187 | 1.17335261 | 0.1 | 1.13816411 | 1.1378446 | 0.0 | 0 |
| 4 | 3.688649698 | 3.69022839 | 0.0 | 2.01400273 | 2.0132905 | 0.0 | 0 |
| 5 | 8.983764227 | 8.98672614 | 0.0 | 3.13928657 | 3.1384141 | 0.0 | 0 |

The deflection and bending moment factors $\alpha$, $\beta$ and $\gamma$ are listed in Tables 1 and 2, where the corresponding results using symplectic method [14] are also listed. Comparison of the results shown in the table illustrates compatibility in spite of the simplicity of the present analysis. The percentage difference between the present results and the symplectic method [14] denoted by $\varepsilon = 100\left|\frac{present - [14]}{[14]}\right|\%$.

A comparison chart of deflection factor $\alpha$ for present study and Symplectic method at the plate center (Fig. 2).

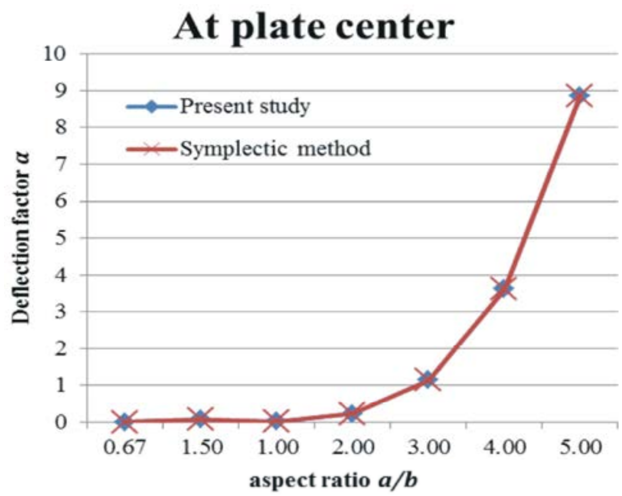


Fig. 2: Chart of deflection factor α for present study and Symplectic method

## CONCLUSION

A method of analysis for calculation of deflection of the opposite edges rectangular via minimum energy concept and substitution of real polynomials at the points over the plate area is presented in this study. It has proved to give accurate results although it is easy, very applicable and fast to be used by engineers in construction sites or manufacturing facilities. Moreover, it can be used under the entire applicable boundary conditions with any in plane and out-plane loadings.

**Appendix:** The current procedure mentioned in the present study is also used in finding the deflection functions for the cases (CCFF) and (CSFF) by using the following functions instead:

$f(x) = c_5(a^2x^2 - 2ax^3 + x_4)$ for (CC) edges and $f(x) = c_5(1.5a^2x^2 - 2.5ax^3 + x^4)$ for (CS) edges in Eq. (3) then apply the same sequence of solution for the new assumed function $w$.

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