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A Special Case of Fourier Spectral Solution for Constant and Variable Coefficient Poisson Equations in 1D and 2D

Pedro Pablo Cárdenas Alzate, Jhon Jairo León Salazar and Fernando Mesa

Department of Mathematics, Universidad Tecnológica de Pereira, Pereira, Colombia

Abstract: A numerical method using Fourier spectral approach is developed to solve the Poissonequation for constant and variable coefficient in 1D and 2D. This method is stable and requires $o(NlogN)$ operations, where$N$ is the number of nodes in the discretization.

Key words: Poisson equation · Fourier Spectral

INTRODUCTION

Spectral methods are a tool for the solution of many problems of PDEs. In the usual formulation, the idea is to represent the solution u by means of a truncated series expansion and to compute spatial derivatives of the function u by differentiation of the series. For periodic functions, Fourier spectral approach is used with the function tabulated at equispaced nodes.

Application of spectral methods, which are based on global expansions into orthogonal polynomials as Legendre or Chebyshev to the solutions of Poisson equations, results in many problems. Here, the Fourier spectral solution gives rise to diagonal matrices and has an exponential rate of convergence but looses accuracy.

Our main goal in this paper is to suggest that Fourier spectral methods be developed through the use of Fourier differentiation matrix to solve the Poisson since this technique works well and we can get machine precision.

We consider the following problem

$$(\beta(x)u_x)_x = f(x) \tag{1}$$

where

$$u:[0,2\pi] \to \mathbb{R}, x \in [0,2\pi], f:[0,2\pi] \to \mathbb{R}$$

and $\beta:[0,2\pi] \to \mathbb{R}$, with u periodic.

Variable Coefficient Case: We assume in this case $\beta(x) = 2 + sin(x)$. The nontrivial functions $u(x)$ and $f(x)$ are $u(x) = sin(2x)$ and consequently $f(x) = -2sin(x) + cos(2x)$. In the domain $[0,2\pi]$, we chose $N = 2^{[1:2:16]} + 1$ number of points, i.e. $2 + 1,8 + 1,2^{16} + 1$ points in total. In Fig. 1 we show the convergence plot as a function of $h = 2\pi/N$.

We can see that Fourier transform does not work well because 2π is actually the same point as 0 and this detail we can definitely hinder the results for the variable coefficient example.

Discontinuous Coefficient Case: Now we assume

$$\beta(x) = \begin{cases} 1, & x \le \frac{\pi}{2} \\ 2, & \frac{\pi}{2} \le x < \frac{3\pi}{2} \\ 1, & x \ge \frac{3\pi}{2} \end{cases}$$

Let $u(x) = sin(2x)$, then $u_{xx} = -4sin(2x)$ and

$$f(x) = sin(2x). \begin{cases} -4, & x \le \frac{\pi}{2} \\ -8, & \frac{\pi}{2} \le x < \frac{3\pi}{2} \\ -4, & x \ge \frac{3\pi}{2} \end{cases}$$

For this problem, we vary $u(x)$ and also we try to multiply in Fourier space, with what was supposed to be a Dirac delta function (which we approximated by a very steep Gaussian curve). This function was applied at the discontinuities.

Correpsonding Author: Pedro Pablo Cárdenas Alzate, Department of Mathematics, Universidad Tecnológica de Pereira, Pereira, Colombia.

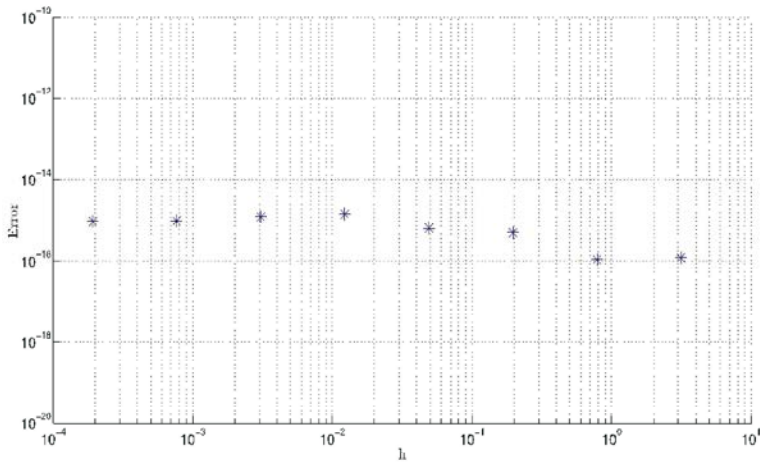

Fig. 1: Convergence plot for $u(x) = sin(x)$ in variable coefficient case

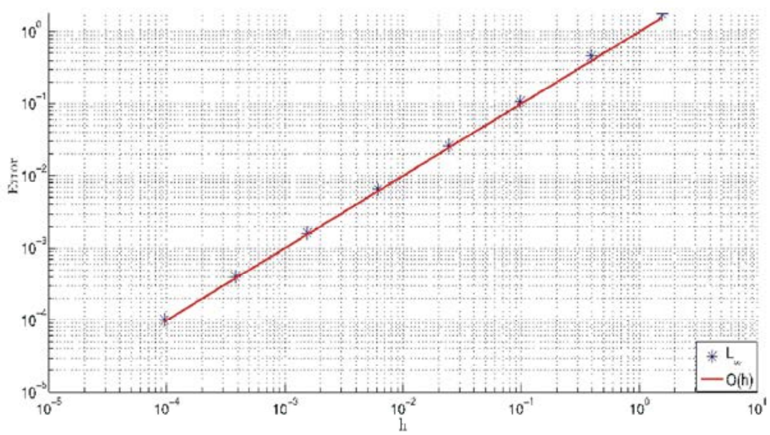

Fig. 2: Convergence plot for $u(x) = sin(2x)$ in discontinuous coefficient case

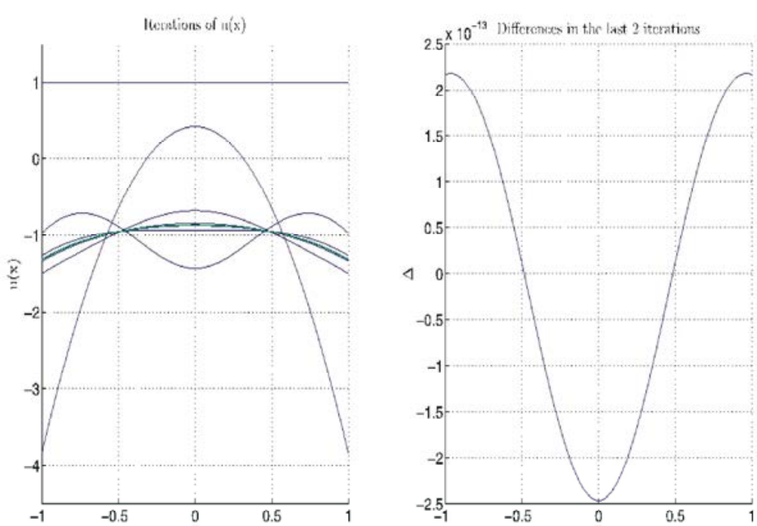

Fig. 3: Successive iterations starting at $u(x) = 1$ converge towards a parabola and difference between the 30th and the 29th iterations is around 3.10^{-13}

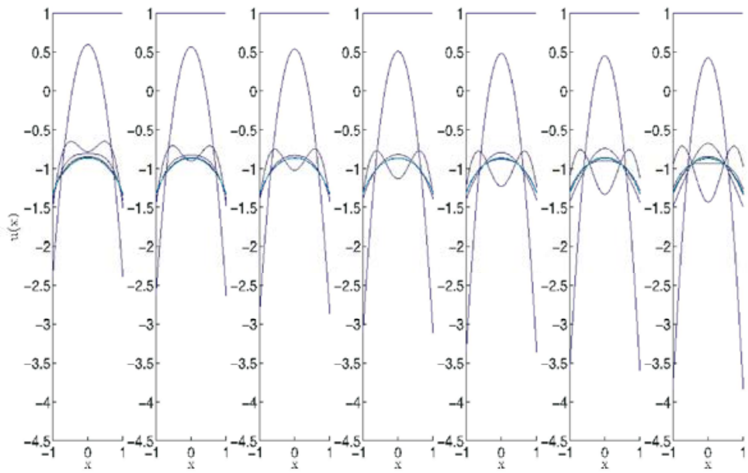

Fig. 4: $u(x)$ vs. x for different values of the parameter α. We can see that starting with an initial condition, there is a downward parabola appearing at the second iteration. All cases of α presented here are converging to the stable solution.

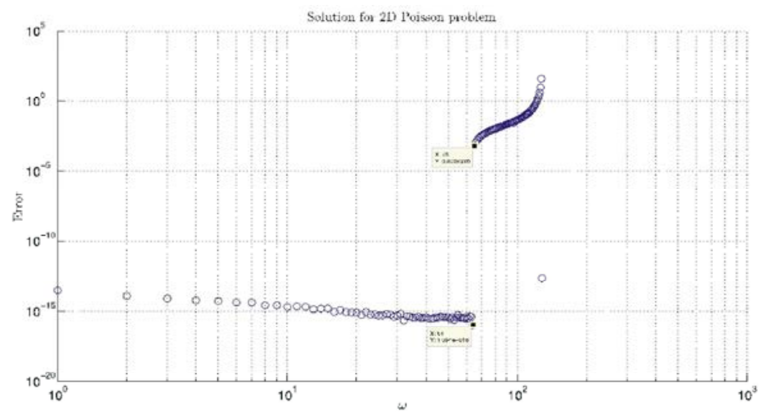

Fig. 5: Error vs. ω for the variable coefficient of the 2D Poisson equation

Therefore, we can see that work quite nicely is applying a constant of integration, after integrating the first time. Then, for $u(x) = sin(2x)$ a constant of integration (C_i) defined as

$$C_i = \cdot \begin{cases} 1, & x \leq \frac{\pi}{2} \\ -1, & \frac{\pi}{2} \leq x < \frac{3\pi}{2} \\ -1, & x \geq \frac{3\pi}{2} \end{cases}$$

can give a linear convergence in the plot Error vs. *h*. The results for the convergence plot are shown in Fig. 2.

Non-Linear Case: We assume $\beta = 1$ and $f(x) = e^{u(x)}$, i.e.

$$u_{xx} = e^{u(x)}$$

This expression is equivalent to

$$L = \frac{d}{dx^2}$$

operator. We need to use a fixed-point iteration.

For this problem, we applied an initial guess for the solution $u(x) = 1$. With 2^{10} points in the interval [-1,1], u converges rapidly to the stable solution shown in Fig. 3. In less than 5 steps, the error is minimal and between the 29th and the 30th iterations, the difference is less than 5.10^{-13} (Fig. 4).

Now, we implemented the stabilization technique

$$u^{n+1} = \alpha u^* + (1 - \alpha)u^n$$

where u^* is the value of u calculated from the iteration, so varying the parameter α, the number of steps required for the error to be minimal changes. In Fig. 5 we can see that for the values of $\alpha < 0.55$, the number of steps would be an error of around 10^{15} is 50. For larges values, this number of steps decreases, reaching a minimum for $\alpha = 0.85$. Only 23 iterations are required to give the same error.

Therefore, when calculating u^{n+1}, we need a balance between the previous solutions, u^n and the calculated value $u*$. Fig. 4 shows the impact of the parameter α on the successive iterations. For example, for $\alpha < 0.85$, the system is *over-relaxed.* Starting at $u(x) = 1$, the solution converges to the parabola but slowly. There is less of an oscillation present.

Now, past $\alpha = 0.85$, the system is *under-relaxed;* large values of $\alpha \in [0.85,1]$ produce oscillations of higher amplitude, which converge slower to the solution as α increases.

We can see in Fig.4 that starting with an initial condition $u(x)$, there is a downward parabola appearing at the second iteration. Third iteration is differently in all cases (oscillating with a different amplitude). All cases of α presented here are converging to the stable solution.

Therefore we can say that when searching for the optimal value of α, we should look for the solution and see what is the result of various iterations. In fact α should be chosen in away that optimizes these results: the outcome of the steps should be close enough to the solution, maybe oscillate, but with low amplitudes.

Fourier-spectral Solution for Constant and Variable Coeffient Poisson Equation in 2D: We consider the following problem on the flat 2-torus $\mathbf{T}^2$,

$$\nabla . (\beta \nabla u) = f$$

where u: $[0,2\pi]^2 \to \mathbb{R}$, f:$[0,2\pi]^2 \to \mathbb{R}$, β:$[0,2\pi]^2 \to \mathbb{R}$, with $\beta > 0$ and we assume that $f(x) = sin(\omega x)cos(\omega y)$ with $\omega \in \mathbb{N}$.

The solution to this problem is

$$u(x) = -\frac{1}{\omega^2} sin(\omega x) sin(\omega y)$$

Taking 128 points on both x and y domain, we can use the Fourier Differentiation Matrix to solve the Poisson Equation. In Fig.5 we show how the solution is converging to real solution. For $\omega \in [1:1:64]$, the technique works well and we can get machine precision. We can see how, for $\omega > 65$, the Nyquist frequency plays a role and the error is very large.

Constant Coefficient Case: We assume $\beta = 1$. In this case we solved this equation for u by using Fourier-Spectral approach and we can see in Fig. 5 the L^∞ norm of the error as a function of $\omega = \{1,2, ... ,64\}$ for a fixed number of points, $Nx = Ny = 128$.

Discontinuous Coefficient Case: We assume in this case $\beta = 4 + cos(x) + sin(y)$, $\omega = 3$. We can note that it is in general not possible to use the same approach as in the 1D variable coefficient case.

Here we can do a little analysis before coding this problem.

$$f = \nabla \cdot (\beta \nabla u) = \nabla \beta \cdot \nabla u + \beta \nabla \cdot \nabla u$$

$$= \beta_x u_x + \beta_y u_y + \beta u_{xx} + \beta u_{yy}$$

where the indices represent the respective derivatives of the functions. In Fourier space we can write

$$\hat{f}_{k,l} = \sum_{m,n} (\hat{\beta}_x)_{k-m,l-n} (\hat{u}_x)_{m,n} +$$

$$\sum_{m,n} (\hat{\beta}_y)_{k-m,l-n} (\hat{u}_y)_{m,n} +$$

$$\sum_{m,n} (\hat{\beta})_{k-m,l-n} (\hat{u}_{xx})_{m,n} +$$

$$\sum_{m,n} (\hat{\beta})_{k-m,l-n} (\hat{u}_{yy})_{m,n}$$

where m,n are the indices of the Fourier transform of f and k,l are the respective convolution

$$= \sum_{m,n} i(k-m)(\hat{\beta})_{k-m,l-n} im(\hat{u})_{m,n}$$

$$+ \sum_{m,n} i(l-n)(\hat{\beta})_{k-m,l-n} in(\hat{u})_{m,n}$$

$$+ \sum_{m,n} -m^2(\hat{\beta})_{k-m,l-n} (\hat{u})_{m,n} +$$

$$\sum_{m,n} -n^2(\hat{\beta})_{k-m,l-n} (\hat{u})_{m,n} =$$

$$\sum_{m,n} (\hat{\beta})_{k-m,l-n} (\hat{u})_{m,n} ((ik - im)im +$$

$(il - in)in - m^2 - n^2) = -(km + ln)$

$$\cdot \sum_{m,n} (\hat{\beta})_{k-m,l-n} (\hat{u})_{m,n}$$

In space Fourier, the convolution very easily a multiplication. After computing this matrix, $\hat{u}$ can be obtained as such

$$\hat{u}_{m,n} = -\left((km + ln) \sum_{m,n} (\hat{\beta})_{k-m,l-n}\right)$$

$$\backslash \hat{f}_{k,l}$$

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