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A New Optimized Runge-Kutta-Nyström Method to Solve Oscillation Problems

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Abstract: In this article, a new Runge-Kutta-Nyström method is derived. The new RKN method has zero phase-lag, zero amplification error and zero first derivative of phase-lag. This method is basically based on the sixth algebraic order Runge-Kutta-Nyström method, which has proposed by Dormand, El-Mikkawy and Prince. Numerical illustrations show that the new proposed method is much efficient as compared with other Runge-Kutta-Nyström methods in the scientific literature, for the numerical integration of oscillatory problems.

Key words: Runge-Kutta-Nyström method · Phase-lag · Amplification error · Derivative · Oscillatory problems

INTRODUCTION

In this article, we deal with the numerical method for solving special second order ordinary differential equations (ODEs) of the form

$$y'' = f(x, y), \quad y(x_0) = y_0, \quad y'(x_0) = y_0' \tag{1}$$

for which their solutions they are known are oscillating. This type of problem occurs in several of applied fields such as quantum mechanics, electronics physical chemistry, molecular dynamics, astronomy, chemical physics and control engineering. In the last decade, many researchers developed methods with minimal phase-lag or phase-lag of order infinity [1-9]. Moreover several methods derived with nullification of phase-lag, amplification factor and phase-lag's derivative for the numerical integration of ordinary differential equations (ODEs) with oscillatory solutions [10-18]. Brusa and Nigro [19] suggested the definition of phase-lag of a method. Several authors have been studied the phase-lag of numerical methods for solving (1) (Houwen and Sommeijer [20, 21] and Thomas [22]). Papadopoulos and Simos [23] suggested a Runge-Kutta-Nyström method with invalidation of phase-lag, amplification factor together with the invalidation of their integrals.

The main goal of this article is to derive an explicit Runge-Kutta-Nyström method with sixth algebraic order, phase-lag of order infinity, first derivative of phase-lag of order infinity and amplification error of order infinity. The new RKN method is based on the parameters of the well-known sixth algebraic order of Dormand, El-Mikkawy and Prince Runge-Kutta-Nyström method [24] with FSAL (first stage as last) property. This means that a RKN method reduces the number of function evaluations.

Phase-Lag Properties of Rkn Method: The general form of Runge-Kutta-Nyström method with *S*-stage for solving second order ordinary differential equations ODEs (1) can be expressed as follows:

$$y_{n+1} = y_n + hy_n' + h^2 \sum_{i=1}^{s} b_i\, k_i,$$

$$y_{n+1}' = y_n' + h \sum_{i=1}^{s} b_i'\, k_i, \tag{2}$$

where,

$$k_i = f(x_n + c_i h, y_n + hc_i y_n' + h^2 \sum_{j=1}^{i-1} a_{ij}\, k_j), i = 1,2, \ldots, s. \tag{3}$$

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In particular $c_1 = 0$ for explicit Runge-Kutta-Nyström method and for an FSAL explicit Runge-Kutta-Nyström method $c_s = 1$ and $a_{sj} = bi$ for $j = i$.

To derived the new RKN method, we consider the test equation.

$$y''(x) = -w^2 y(x) \quad w > 0. \tag{4}$$

By substituting $f(x, y) = -w^2 y$ into equations (2), which lead to the following numerical solution.

$$\begin{bmatrix} y_n \\ hy'_n \end{bmatrix} = D^n \begin{bmatrix} y_0 \\ hy_0 \end{bmatrix}, \quad D = \begin{bmatrix} A(z^2) & B(z^2) \\ A'(z^2) & B'(z^2) \end{bmatrix}, z = wh, \tag{5}$$

where A,B,A' and B' are polynomial in expression of z^2 and are fully determined by the coefficients of RKN method (2). The characteristic equation of matrix D is.

$$r^2 - trace(D)r + det(D) = 0. \tag{6}$$

The solution of equation (4) is defined by.

$$y(x_n) = \varrho_1 \left[e^{(iz)}\right]^n + \varrho_2 \left[e^{(-iz)}\right]^n, \tag{7}$$

where

$$\varrho_{1,2} = \frac{1}{2}\left[y_0 \pm \frac{(iy'_0)}{v}\right] \quad or \quad \varrho_{1,2} = |\varrho|\, e^{(\pm i\chi)}. \tag{8}$$

After substituting equation (11) into equation (10), we get

$$y(x_n) = 2|\varrho| \cos(\chi + nz). \tag{9}$$

Therefore, we suppose that the eigenvalues of matrix D are ϑ_1, ϑ_2 and they are denoted as the amplification error of the RKN method. The resultant eigenvectors are $[1,v_1]^T$, $[1,v_2]^T$ where $v_i = A'/(\vartheta_i - B')$, i=1,2. They yield the numerical solution of (4) as follows:

$$y_n = c_1 \vartheta_1^n + c_2 \vartheta_2^n, \tag{10}$$

where,

$$c_1 = -\frac{v_2 y_0 - hy'_0}{v_1 - v_2}, \qquad c_2 = \frac{v_1 y_0 - hy'_0}{v_1 - v_2}. \tag{11}$$

If ϑ_1 and ϑ_2 are complex conjugate, thus $c_{1,2} = |c|\, e^{(\pm w)}$ and $\vartheta_{1,2} = |\vartheta|\, e^{(\pm w)}$. Now substituting into equation (10), we obtain.

$$y_n = 2|c||\vartheta| \cos(w + n\phi). \tag{12}$$

The following definition which yields from numerical solution (12) and exact solution (9).

Definition 1: (Phase-lag [21]) By applying the Runge-Kutta-Nyström RKN method (2) to the test equation (4), therefore we can define the phase-lag as follows:

$$\psi(z^2) = z - \cos^{-1}\left(\frac{trace(D)}{2\sqrt{det(D)}}\right). \tag{13}$$

The RKN method is said to have phase-lag (dispersion error) of order q if $\psi(z^2) = O(h^{q+1})$. Moreover, the quantity $\alpha(z^2) = 1 - \sqrt{det(D)}$ is called amplification error (dissipation error). Also the method is said to have dissipative of order p if $\alpha(z^2) = O(h^{p+1})$.

we will refer that

$$S(z^2) = trace(D) = A(z^2) + B'(z^2),$$

$$T(z^2) = det(D) = A(z^2)B'(z^2) - A'(z^2)B(z^2), \tag{14}$$

Thus

$$\psi(z) = z - \cos^{-1}\left(\frac{S(z^2)}{2\sqrt{T(z^2)}}\right), \quad \alpha(z) = 1 - \sqrt{T(z^2)}. \tag{15}$$

In the situation of phase-lag of order infinity, we obtain the following definition.

Definition 2: (phase-lag of order infinity [16]): To get phase-lag of order infinity, we must satisfy the relation $\psi(z) = z - \cos^{-1}\left(\frac{S(z^2)}{2\sqrt{T(z^2)}}\right) = 0$.

The RKN method is said to have zero dissipative (zero amplification error) if $a(z) = 0$ at a point z, which produces

$$\alpha(z) = 1 - \sqrt{T(z^2)} = 0 \Rightarrow T(z^2) = 1. \tag{16}$$

From equation (16) and Definition 2, we obtain the following remarks.

Remark 3 [23]: To obtain the phase-lag of order infinity and amplification error of order infinity, we must satisfy the following relations:

Table 1: The Dormand, El-Mikkawy and Prince RKN Method

0					
$\frac{1}{10}$	$\frac{1}{200}$				
$\frac{3}{10}$	$-\frac{1}{2200}$	$\frac{1}{22}$			
$\frac{7}{10}$	$\frac{637}{6600}$	$-\frac{7}{100}$	$\frac{7}{33}$		
$\frac{17}{25}$	$\frac{225437}{1968750}$	$-\frac{30073}{281250}$	$\frac{65569}{281250}$	$-\frac{9367}{984375}$	
1	$\frac{151}{2142}$	$\frac{5}{116}$	$\frac{385}{1368}$	$\frac{55}{168}$	$-\frac{6250}{28101}$
	$\frac{151}{2142}$	$\frac{5}{116}$	$\frac{385}{1368}$	$\frac{55}{168}$	$-\frac{6250}{28101}$ $\quad 0$
	$\frac{151}{2142}$	$\frac{25}{522}$	$\frac{275}{684}$	$\frac{275}{252}$	$-\frac{78125}{112404}$ $\quad \frac{1}{12}$

$$S(z^2) = 2\cos(z),$$

$$T(z^2) = 1.$$

Remark 4: To construct the optimized RKN method by using the technique of zero phase-lag and zero phase lag's derivative together with zero amplification error, we must satisfy the following conditions:

$$S(z^2) = 2\cos(z),$$

$$S'(z^2) = -2\sin(z), \tag{17}$$

$$T(z^2) = 1.$$

Construction of the Optimized Rkn Method: In this section, we derive an optimized RKN method by nullifying the phase-lag, the first derivative of phase-lag and the amplification error, which is based on the Dormand, El-Mikkawy and Prince Runge-Kutta-Nyström method (Table 1) of sixth algebraic order with FSAL technique (first stage as last), thus the optimized RKN method in fact utilizes five stage per step for the function evaluations. In order to derive the optimized RKN method. Firstly, we set b'_4, b'_5, and b'_6, as free parameters while all other coefficients are the same as in Table 3. Secondly, we compute the polynomials A,B,A and B' in terms of optimized Runge-Kutta-Nyström RKN parameters. After that, by substituting these polynomials in equation (14) we obtain the expressions of $S(z^2)$ and $S(z^2)$. In addition, we find the derivative of $S(z^2)$. Finally, from Remark 4 we have the system of three equations as follows:

$$\begin{aligned}
&2 - \frac{1}{9801000} z^{10} + \frac{559}{23760000} z^8 - \frac{1}{720} z^6 + \frac{1}{24} z^4 - \frac{1}{2} z^2 - \frac{1}{490050} b'_6 z^{10} \\
&+ \left(\frac{239}{1188000} b'_6 - \frac{9367}{1020937500} b'_5\right) z^8 + \left(-\frac{1}{120} b'_6 - \frac{7}{7260} b'_4 - \frac{159239}{309375000} b'_5\right) z^6 \\
&+ \left(\frac{1}{6} b'_6 + \frac{63}{1100} b'_4 + \frac{5}{2736} + \frac{493}{9375} b'_5\right) z^4 + \left(-\frac{7}{10} b'_4 - b'_6 - \frac{4975}{39672}\right. \\
&\left. - \frac{17}{25} b'_5\right) z^2 - 2\cos(z) = 0,
\end{aligned} \tag{18}$$

$$\begin{aligned}
&-\frac{1}{980100} z^9 + \frac{559}{2970000} z^7 - \frac{1}{120} z^5 + \frac{1}{6} z^3 - z - \frac{1}{49005} b'_6 z^9 + \left(\frac{1912}{1188000} b'_6\right. \\
&\left. - \frac{74936}{1020937500} b'_5\right) z^7 + \left(-\frac{6}{120} b'_6 - \frac{42}{7260} b'_4 - \frac{955434}{309375000} b'_5\right) z^5 + \left(\frac{4}{6} b'_6\right. \\
&\left. + \frac{252}{1100} b'_4 + \frac{20}{2736} + \frac{1972}{9375} b'_5\right) z^3 + \left(-\frac{14}{10} b'_4 - 2b'_6 - \frac{9950}{39672} - \frac{34}{25} b'_5\right) z \\
&+ 2\sin(z) = 0,
\end{aligned} \tag{19}$$

$$\begin{aligned}
&1 - \frac{165185}{1573656} z^2 + \frac{1698967}{1475923680000} z^8 + \frac{338029}{17015625000} z^8 b'_5 - \frac{271}{726000} z^6 b'_4 \\
&- \frac{25123}{38671875} z^6 b'_5 - \frac{29}{6600} z^4 b'_4 - \frac{33}{6250} z^4 b'_5 + \frac{3}{10} z^2 b'_4 + \frac{8}{25} z^2 b'_5
\end{aligned}$$

$$+\frac{17}{245025000}z^{10}b_4' - \frac{149011}{2450250000000}z^{10}b_5' + \frac{307}{26136000}z^8 b_4'$$

$$-\frac{916019}{11567551842000}z^{10} - \frac{927}{20982080}z^6 + \frac{63743}{56651616} - 1 = 0 \tag{20}$$

Solving this system simultaneously we obtained the coefficients of b'_4, b'_5 and b'_6, which are totally dependent on z, where z is the product of the frequency of the method w and the step size h. The expressions for b'_4, b'_5 and b'_6 are too complicated. As the small value of z_i therefore we use the following Taylor series expressions

$$b_4' = \frac{275}{252} - \frac{8053}{2235618}z^4 + \frac{122085839893}{203999503752000}z^6 - \frac{283077421408516909}{380054135482532280000 0}z^8$$

$$+\frac{1379062961251270743082584 1}{102273391308976315367940000000}z^{10}$$

$$-\frac{4142309519821164772098205032 1121}{1749828325495339379160025155000000000}z^{12} + \cdots$$

$$b_5' = -\frac{78125}{112404} + \frac{40265}{11923296}z^4 - \frac{79297273613}{142799652626400}z^6$$

$$+\frac{361892175170188609}{5067388473100430400000}z^8 - \frac{4783585359212839367404447 57}{36818420871231473532458400000000}z^{10}$$

$$+\frac{6927054685059891803945548508 2301}{3062199569616843913530044021250000000}z^{12} + \cdots$$

$$b_6' = \frac{1}{12} + \frac{8053}{35769888}z^4 - \frac{14462381887}{407999007504000}z^6$$

$$+\frac{361781162875243}{125637730738027200000}z^8 - \frac{2075714685289242738277861}{3347129170111952139314400000000}z^{10}$$

$$+\frac{4665336931079526965417708 3521}{4124174504534469917212180500000000000}z^{12} + \cdots$$

Numerical Experiments: In this section, to test the efficiency of the optimized Runge-Kutta-Nyström RKN method. We will solve six oscillatory problems. The numerical results obtained are compared with the well-known methods which are chosen from the scientific literature. We use in the numerical comparisons the criteria based on computing the maximum error in the solution (Max Error = max ($|y(t_n) - y_n|$)) which is equal to the maximum between absolute errors of the true solutions and the computed solutions. Figures 1–6 show the efficiency curves of log_{10}(Max Error) against the computational effort measured by (CPU Time Second). The following methods are used in the comparison:

- ORKN6: The new optimized sixth-order Runge-Kutta-Nyström RKN method derived in section 3 in this paper.
- RKN6: The high order method of embedded RKN 6(4)6FM method of Dormand, El-Mikkawy and Prince [24].
- RKN6A: The Runge-Kutta-Nyström RKN formula of Albrecht [25].
- RKN6MR: The high order method of Embedded Runge-Kutta-Nyström RKN method, constructed by El-Mikkawy and Rahmo [26].

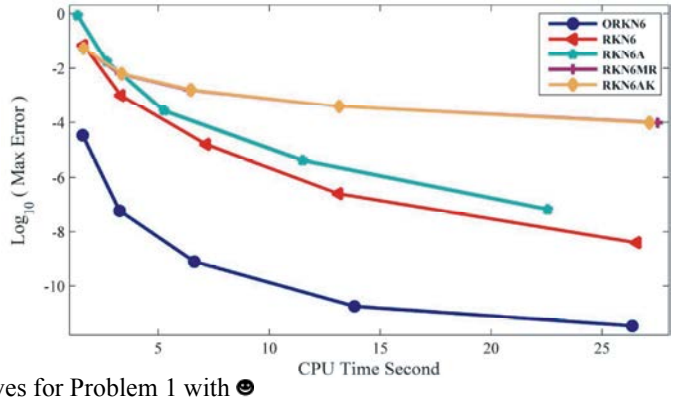

Fig. 1: The efficiency curves for Problem 1 with ☻

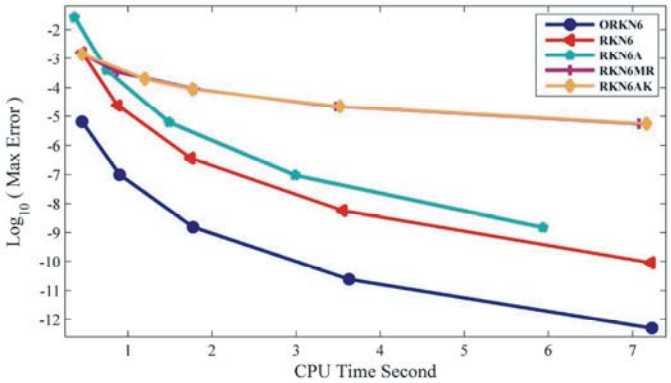

Fig. 2: The efficiency curves for Problem 2 with $h = 0.8/2^i$, $i = 0,..., 4$.

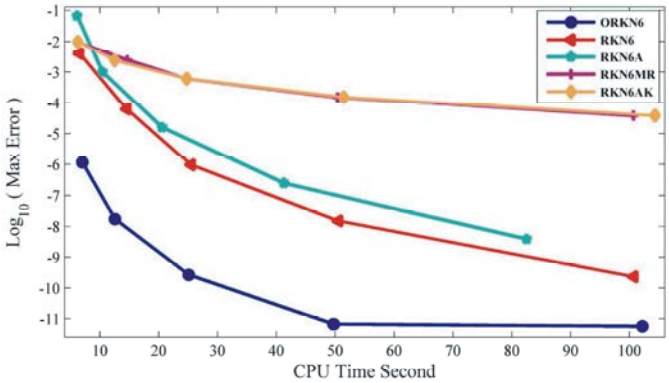

Fig. 3: The efficiency curves for Problem 3 with ☻

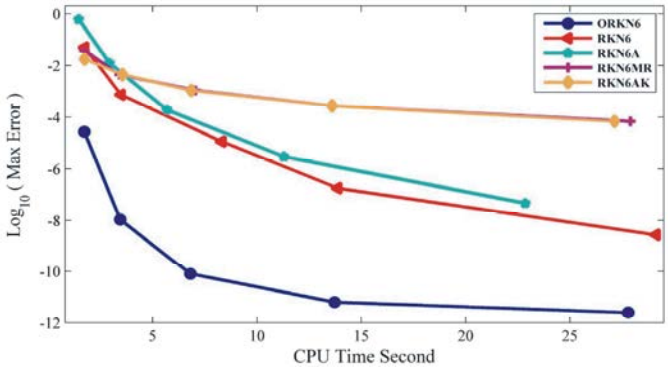

Fig. 4: The efficiency curves for Problem 4 with $h = 0.1/2^i$, $i = 0,..., 4$.

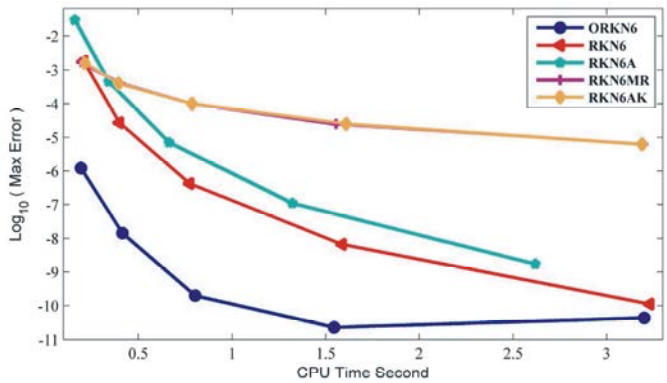

Fig. 5: The efficiency curves for Problem 5 with $h = 0.8/2^i$, $i = 0,..., 4$.

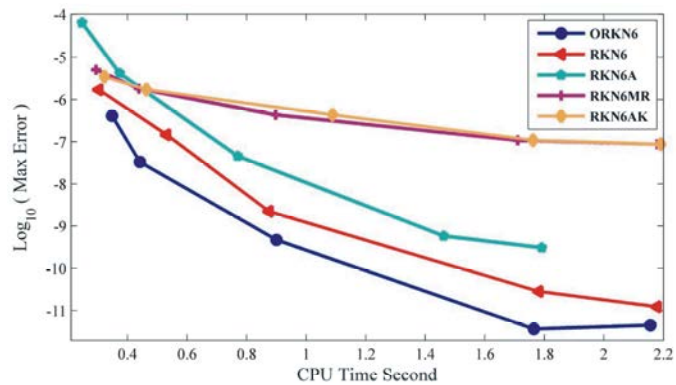

Fig. 6: The efficiency curves for Problem 6 with $h = 0$.

- RKN6AK: The high order method of optimized embedded RKN 6(4) method derived by Anastassi and Kosti [27].

Problem 1: (Inhomogeneous equation studied by Simos [6]).

$$y'' = -100y + 99\sin(x), \quad y(0) = 1, \quad y'(0) = 11.$$

The frequency is $w = 10$ and the exact solution is

$$y(x) = \cos(10x) + \sin(10x) + \sin(x)$$

Problem 2: (An almost periodic orbit problem given in Stiefel and Bettis [28]).

$$y_1'' + y_1 = 0.001\cos(x), \quad y(0) = 1, \quad y'(0) = 0.$$

$$y_2'' + y_2 = 0.001\sin(x), \quad y(0) = 0, \quad y'(0) = 0.9995.$$

The frequency is $w = 1$ and the exact solutions are.

$$y_1(x) = \cos(x) + 0.0005x\sin(x),$$

$$y_2(x) = \sin(x) - 0.0005x\cos(x).$$

Problem 3: (Inhomogeneous linear system studied by Franco [29]).

$$y_1'' + \frac{101}{2}y_1 - \frac{99}{2}y_2 = \frac{93}{2}\cos(2x) - \frac{99}{2}\sin(2x), \quad y_1(0) = 0, \quad y_1'(0) = -10.$$

$$y_2'' - \frac{99}{2}y_1 + \frac{101}{2}y_2 = \frac{93}{2}\sin(2x) - \frac{99}{2}\cos(2x), \quad y_1(0) = 1, \quad y_1'(0) = 12.$$

The frequency is $w = 10$ and the exact solutions are.

$$y_1(x) = -\cos(10x) - \sin(10x) + \cos(2x),$$

$$y_2(x) = \cos(10x) + \sin(10x) + \sin(2x).$$

Problem 4: (Harmonic oscillator studied by Anastassi and Kosti [27]).

$$y'' = -100y, \quad y(0) = 1, \quad y'(0) = 0.$$

The frequency is $w = 10$ and the exact solution is $y(x) = \cos(10x)$

Problem 5: (Inhomogeneous equation studied by Al-Khasawneh *et al*. [30]).

$$y'' = -y + x, \quad y(0) = 1, \quad y'(0) = 2.$$

The frequency is $w = 1$ and the exact solution is $y(x) = \sin(x) + \cos(x) + x$.

Problem 6: (Duffing equation [31]).

$$y'' = -y + y^3 + 0.002\cos(1.01x), \quad y(0) = 0.200426728067, \quad y'(0) = 0.$$

The frequency is $w = 1$, and the exact solution is.

$$y(x) = 0.200179477536\cos(1.01x) + 0.000246946143\cos(3.03x)$$

$$+ 0.000000304014\cos(5.05x) + 0.000000000374\cos(7.07x).$$

CONCLUSION

In the present paper, we have derived the new optimized Runge-Kutta-Nyström RKN method with vanishing the phase-lag, the first derivative of phase-lag and amplification error. Which is based on the Runge-Kutta-Nyström RKN method of sixth algebraic order, derived by Dormand, El-Mikkawy and Prince [24]. From the numerical results we conclude that, the new optimized RKN method is computationally more efficient and effective in solving special second order ODEs with oscillating solutions and outperformed the well-known RKN methods in the scientific literature.

REFERENCES

1. Chawla, M.M. and P.S. Rao, 1984. A Noumerov-type method with minimal phase-lag for the integration of second order periodic initial-value problems, Journal of Computational and Applied Mathematics, 11(3): 277-281.
2. Chawla, M.M. and P.S. Rao, 1986. A Noumerov-type method with minimal phase-lag for the integration of second order periodic initial-value problems. II: Explicit method, Journal of Computational and Applied Mathematics, 15(3): 329-337.

3. Chawla, M.M. and P.S. Rao, 1987. An explicit sixth-order method with phase-lag of order eight for $y'' = f(t, y)$, Journal of Computational and Applied Mathematics, 17(3): 365-368.
4. Simos, T.E., E. Dimas and A.B. Sideridis, 1994. A Runge-Kutta-Nyström for the numerical integration of special second-order periodic initial-value problems, Journal of Computational and Applied Mathematics, 51(3): 317-326.
5. Simos, T.E., 1993. A Runge-Kutta-Fehlberg method with phase-lag of order infinity for initial-value problems with oscillating solution, Computers and Mathematics with Applications, 25(6): 95-101.
6. Simos, T.E., 1993. Runge-Kutta-Nyström interpolants for the numerical integration of special second-order periodic initial-value problems, Computers and Mathematics with Applications, 26(12): 7-15.
7. Simos, T.E., 1998. Some modified Runge-Kutta methods for the numerical solution of initial-value problems with oscillating solutions, Journal of Scientific Computing, 13(1): 51-63.
8. Simos, T.E. and J.V. Aguiar, 2001. A modified Runge-Kutta method with phase-lag of order infinity for the numerical solution of the Schrödinger equation and related problems, Computers and Chemistry, 25(3): 275-281.
9. Van De Vyver, H., 2007. A symplectic Runge-Kutta-Nyström method with minimal phase- lag, Physics Letters A, 367(1): 16-24.
10. Ahmad, S.Z., F. Ismail, N. Senu and M. Suleiman, 2013. Zero-dissipative phase-fitted hybrid methods for solving oscillatory second order ordinary differential equations, Applied Mathematics and Computation, 219(19): 10096-10104.
11. Alolyan, I. and T.E. Simos, 2010. High algebraic order methods with vanished phase-lag and its first derivative for the numerical solution of the Schrödinger equation, Journal of Mathematics Chemistry, 48(4): 925-958.
12. Alolyan, I. and T.E. Simos, 2011. A family of ten-step methods with vanished phase-lag and its first derivative for the numerical solution of the Schrödinger equation, Journal of Mathematics Chemistry, 49(9): 1843-1888.
13. Konguetsof, A., 2011. A hybrid method with phase-lag and derivatives equal to zero for the numerical integration of the Schrödinger equation, Journal of Mathematics Chemistry, 49(7): 1330-1356.
14. Kosti, A.A., Z.A. Anastassi and T.E. Simos, 2010. An optimized explicit Runge-Kutta method with increased phase-lag order for the numerical solution of the Schrödinger equation and related problems, Journal of Mathematics Chemistry, 47(1): 315-330.
15. Papadopoulos, D.F., Z.A. Anastassi and T.E. Simos, 2010. A modified phase-fitted and amplification-fitted Runge-Kutta-Nyström method for the numerical solution of the radial Schrödinger equation, Journal of Molecular Modeling, 16(8): 1339-1346.
16. Papadopoulos, D.F. and T.E. Simos, 2013. A Modified Runge-Kutta-Nyström Method by using Phase Lag Properties for the Numerical Solution of Orbital Problems, Applied Mathematics and Information Sciences, 7(2): 433-437.
17. Senu, N., F. Ismail, S.Z. Ahmad and M. Suleiman, 2015. Optimized Hybrid Methods for Solving Oscillatory Second Order Initial Value Problems, Discrete Dynamics in Nature and Society, Volume 2015, Article ID 217578, 11pages.doi.org/10.1155/ 2015/217578.
18. Simos, T.E., 2009. A new Numerov-type method for the numerical solution of the Schrödinger equation, Journal of Mathematics Chemistry, 46(3): 981-1007.
19. Brusa, L. and L. Nigro, 1980. A one-step method for direct integration of structural dynamic equations, International Journal for Numerical Methods in Engineering, 15(5): 685-699.
20. Van der Houwen, P.J. and B.P. Sommeijer, 1987. Explicit Runge-Kutta-Nyström methods with reduced phase errors for computing oscillating solutions, SIAM Journal on Numerical Analysis, 24(3): 595-617.
21. Van Der Houwen, P.J. and B.P. Sommeijer, 1989. Diagonally implicit Runge-Kutta- Nyström methods for oscillatory problems, SIAM Journal on Numerical Analysis, 26(2): 414-429.
22. Thomas, R.M., 1984. Phase Properties of higher order, almost P-stable formulae, BIT Numerical Mathematics, 24(2): 225-238.
23. Papadopoulos, D.F. and T.E. Simos, 2011. A New Methodology for the Construction of Optimized Runge-Kutta-Nyström Methods, International Journal of Modern Physics C, 22(6): 623-634.
24. Dormand, J.R., M.E.A. El-Mikkawy and P.J. Prince, 1987. Families of Runge-Kutta-Nyström formulae, IMA Journal of Numerical Analysis, 7(2): 235-250.
25. Albercht, J., 1955. Beitrage zum Runge-Kutta-Verfahren, ZAMM-Journal of Applied Mathematics and Mechanics, 35(3): 100-110.

26. El-Mikkawy, M. and E. Rahmo, 2003. A new optimized non-FSAL embedded Runge- Kutta-Nyström algorithm of orders 6 and 4 in six stages, Applied Mathematics and Computation, 145(1): 33-43.
27. Anastassi, Z.A. and A.A. Kosti, 2015. A 6(4) optimized embedded Runge-Kutta-Nyström pair for the numerical solution of periodic problems, Journal of Computational and Applied Mathematics, 275: 311-320.
28. Stiefel, E. and D.G. Bettis, 1969. Stabilization of Cowells methods, Numerische Mathematik, 13(2): 154-175.
29. Franco, J.M., 2003. A 5(3) pair of explicit ARKN methods for the numerical integration of perturbed oscillators, Journal of Computational and Applied Mathematics, 161(2): 283-293.
30. Al-Khasawneh, R.A., F. Ismail and M. Suleiman, 2007. Embedded diagonally implicit Runge-Kutta-Nyström 4(3) pair for solving special second-order IVPs, Applied Mathematics and Computation, 190(2): 1803-1814.
31. Anastassi, Z.A. and T.E. Simos, 2004. Special Optimized Runge-Kutta Methods for IVPs with Oscillating Solutions, International Journal of Modern Physics C, 15(1): 1-15.