A New Optimized Runge-Kutta-Nyström Method to Solve Oscillation Problems

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Abstract: In this article, a new Runge-Kutta-Nyström method is derived. The new RKN method has zero phase-lag, zero amplification error and zero first derivative of phase-lag. This method is basically based on the sixth algebraic order Runge-Kutta-Nyström method, which has proposed by Dormand, El-Mikkawy and Prince. Numerical illustrations show that the new proposed method is much efficient as compared with other Runge-Kutta-Nyström methods in the scientific literature, for the numerical integration of oscillatory problems.

Key words: Runge-Kutta-Nyström method · Phase-lag · Amplification error · Derivative · Oscillatory problems

INTRODUCTION

In this article, we deal with the numerical method for solving special second order ordinary differential equations (ODEs) of the form

\[ y'' = f(x,y), \quad y'(x_0) = y_0, \quad y'(x_0) = y_0' \]  

(1)

for which their solutions they are known are oscillating. This type of problem occurs in several of applied fields such as quantum mechanics, electronics physical chemistry, molecular dynamics, astronomy, chemical physics and control engineering. In the last decade, many researchers developed methods with minimal phase-lag or phase-lag of order infinity [1-9]. Moreover several methods derived with nullification of phase-lag, amplification factor and phase-lag’s derivative for the numerical integration of ordinary differential equations (ODEs) with oscillatory solutions [10-18]. Brusa and Nigro [19] suggested the definition of phase-lag of a method. Several authors have been studied the phase-lag of numerical methods for solving (1) (Houwen and Sommeijer [20, 21] and Thomas [22]). Papadopoulos and Simos [23] suggested a Runge-Kutta-Nyström method with invalidation of phase-lag, amplification factor together with the invalidation of their integrals.

The main goal of this article is to derive an explicit Runge-Kutta-Nyström method with sixth algebraic order, phase-lag of order infinity, first derivative of phase-lag of order infinity and amplification error of order infinity. The new RKN method is based on the parameters of the well-known sixth algebraic order of Dormand, El-Mikkawy and Prince Runge-Kutta-Nyström method [24] with FSAL (first stage as last) property. This means that a RKN method reduces the number of function evaluations.
In particular, \( c_i = 0 \) for explicit Runge-Kutta-Nyström method and for an FSAL explicit Runge-Kutta-Nyström method, \( c_i = 1 \) and \( a_i = bi \) for \( j = i \).

To derive the new RKN method, we consider the test equation:

\[
y''(x) = -w^2 y(x), \quad w > 0.
\]  

(4)

By substituting \( f(x, y) = -w^2 y \) into equations (2), which lead to the following numerical solution:

\[
\begin{bmatrix}
    y_n \\
    h y_n
\end{bmatrix} = D^n \begin{bmatrix}
    y_0 \\
    h y_0
\end{bmatrix}, \quad D = \begin{bmatrix}
    A(z^2) & B(z^2) \\
    A'(z^2) & B'(z^2)
\end{bmatrix}, \quad z = wh,
\]

(5)

where \( A, B, A' \) and \( B' \) are polynomial in expression of \( z \) and are fully determined by the coefficients of RKN method (2). The characteristic equation of matrix \( D \) is:

\[
r^2 - \text{trace}(D) r + \det(D) = 0.
\]  

(6)

The solution of equation (4) is defined by:

\[
y(x_n) = \varphi_{1z} e^{i z x} + \varphi_{2z} e^{-i z x},
\]

(7)

where

\[
\begin{align*}
\varphi_{1z} &= \frac{1}{2} \left[ \frac{\lambda y_0}{\lambda} \pm \frac{(\lambda y_0')}{\lambda} \right], \quad \text{or} \quad \varphi_{1z} = |\lambda| e^{i \lambda x},
\varphi_{2z} &= \frac{1}{2} \left[ \frac{\lambda y_0}{\lambda} \pm \frac{(\lambda y_0')}{\lambda} \right].
\end{align*}
\]

(8)

After substituting equation (11) into equation (10), we get

\[
y(x_n) = 2 |\lambda| \cos(\lambda + nx)
\]

(9)

where

\[
c_1 = \frac{\lambda y_0}{\lambda}, \quad c_2 = \frac{\lambda y_0'}{\lambda},
\]

(10)

If \( \varphi_{1z} \) and \( \varphi_{2z} \) are complex conjugate, thus \( c_{1z} = |\lambda| e^{i \lambda x} \) and \( c_{2z} = |\lambda| e^{-i \lambda x} \). Now substituting into equation (10), we obtain:

\[
y_n = \frac{2 |\lambda| |\lambda|}{c} \cos(\lambda + nx).
\]  

(12)

The following definition which yields from numerical solution (12) and exact solution (9).

**Definition 1:** (Phase-lag [21]) By applying the Runge-Kutta-Nyström RKN method (2) to the test equation (4), therefore we can define the phase-lag as follows:

\[
\psi(z^2) = z - \frac{1}{2} \text{trace}(D).
\]

(13)

The RKN method is said to have phase-lag (dispersion error) of order \( q \) if \( \psi(z^2) = O(h^{q+1}) \). Moreover, the quantity \( a(z^2) = 1 - \sqrt{\det(D)} \) is called amplification error (dissipation error). Also the method is said to have dissipative of order \( p \) if \( a(z^2) = O(h^{p+1}) \).

we will refer that

\[
S(z^2) = \text{trace}(D) = A(z^2) + B'(z^2),
\]

\[
T(z^2) = \det(D) = A(z^2)B'(z^2) - A'(z^2)B(z^2).
\]

(14)

Thus

\[
\psi(z) = z - \cos \left( \frac{S(z^2)}{2 \sqrt{T(z^2)}} \right), \quad a(z) = 1 - \sqrt{T(z^2)}.
\]

(15)

In the situation of phase-lag of order infinity, we obtain the following definition.

**Definition 2:** (phase-lag of order infinity [16]): To get phase-lag of order infinity, we must satisfy the relation

\[
\psi(z) = z - \cos \left( \frac{S(z^2)}{2 \sqrt{T(z^2)}} \right) = 0.
\]

The RKN method is said to have zero dissipative (zero amplification error) if \( a(z) = 0 \) at a point \( z \), which produces

\[
a(z) = 1 - \sqrt{T(z^2)} = 0 \Rightarrow T(z^2) = 1.
\]

(16)

From equation (16) and Definition 2, we obtain the following remarks.

**Remark 3 [23]:** To obtain the phase-lag of order infinity and amplification error of order infinity, we must satisfy the following relations:
Remark 4: To construct the optimized RKN method by using the technique of zero phase-lag and zero phase lag’s derivative together with zero amplification error, we must satisfy the following conditions:

\[ S(z^2) = 2 \cos(z), \]

\[ T(z^2) = 1. \]

Construction of the Optimized RKN Method: In this section, we derive an optimized RKN method by nullifying the phase-lag, the first derivative of phase-lag and the amplification error, which is based on the Dormand, El-Mikkawy and Prince Runge-Kutta-Nyström method (Table 1) of sixth algebraic order with FSAL technique (first stage as last), thus the optimized RKN method in fact utilizes five stage per step for the function evaluations. In order to derive the optimized RKN method. Firstly, we set \( b', b'' \), and \( b''' \) as free parameters while all other coefficients are the same as in Table 3. Secondly, we compute the polynomials \( A, B, A' \) and \( B' \) in terms of optimized Runge-Kutta-Nyström RKN parameters. After that, by substituting these polynomials in equation (14) we obtain the expressions of \( S(z^2) \) and \( S(z^2) \). In addition, we find the derivative of \( S(z^2) \). Finally, from Remark 4 we have the system of three equations as follows:

\[ S(z^2) = 2 \cos(z), \quad (17) \]

\[ T(z^2) = 1. \]

\[ S'(z^2) = -2 \sin(z), \quad (18) \]

\[ T'(z^2) = 1. \]

\[ S''(z^2) = -4 \cos(z), \quad (19) \]
Solving this system simultaneously we obtained the coefficients of $b'_\alpha$, $b'_\beta$ and $b'_\gamma$ which are totally dependent on $z$, where $z$ is the product of the frequency of the method $w$ and the step size $h$. The expressions for $b'_\alpha$, $b'_\beta$ and $b'_\gamma$ are too complicated. As the small value of $z$, therefore we use the following Taylor series expressions

$$b'_\alpha = \frac{275}{252} - \frac{8053}{2235618} z + \frac{122085839893}{203990503752000} z^6 - \frac{283077421408516909}{380541354825328000000000} z^8 + \ldots$$

$$b'_\beta = \frac{916019 z^{10}}{11567558142000} - \frac{927}{20982080} z^8 + \frac{6374356651616}{565651616} z - 1 = 0 \quad (20)$$

$$b'_\gamma = \frac{1}{12} + \frac{8053}{3576988} z^6 - \frac{4462381887}{40799007504000} z^6 + \frac{361781162875243}{1256377307802772000000} z^6 - \frac{2075371468528924738277861}{334712917011952139314400000000000} z^{10} + \ldots$$

**Numerical Experiments:** In this section, to test the efficiency of the optimized Runge-Kutta-Nyström RKN method. We will solve six oscillatory problems. The numerical results obtained are compared with the well-known methods which are chosen from the scientific literature. We use in the numerical comparisons the criteria based on computing the maximum error in the solution (Max Error = \( \text{max} (|y(t) - y|) \)) which is equal to the maximum between absolute errors of the true solutions and the computed solutions. Figures 1–6 show the efficiency curves of \( \log_{10} \text{(Max Error)} \) against the computational effort measured by (CPU Time Second). The following methods are used in the comparison:

- ORKN6: The new optimized sixth-order Runge-Kutta-Nyström RKN method derived in section 3 in this paper.
- RKN6A: The Runge-Kutta-Nyström RKN formula of Albrecht [25].
- RKN6MR: The high order method of Embedded Runge-Kutta-Nyström RKN method, constructed by El-Mikkawy and Rahmo [26].
Fig. 1: The efficiency curves for Problem 1 with $\varepsilon$.

Fig. 2: The efficiency curves for Problem 2 with $h = 0.8/2^i$, $i = 0, \ldots, 4$.

Fig. 3: The efficiency curves for Problem 3 with $\varepsilon$.

Fig. 4: The efficiency curves for Problem 4 with $h = 0.1/2^i$, $i = 0, \ldots, 4$. 

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Fig. 5: The efficiency curves for Problem 5 with $h = 0.8/2^i$, $i = 0, \ldots, 4$.

Fig. 6: The efficiency curves for Problem 6 with $h = 0$.

- RKN6AK: The high order method of optimized embedded RKN 6(4) method derived by Anastassi and Kosti [27].

**Problem 1**: (Inhomogeneous equation studied by Simos [6]).

$$y'' = -100y + 99 \sin(x), \quad y(0) = 1, \quad y'(0) = 11.$$

The frequency is $w = 10$ and the exact solution is

$$y(x) = \cos(10x) + \sin(10x) + \sin(x)$$

**Problem 2**: (An almost periodic orbit problem given in Stiefel and Bettis [28]).

$$y_1'' + y_1 = 0.001 \cos(x), \quad y_1(0) = 1, \quad y_1'(0) = 0.$$

$$y_2'' + y_2 = 0.001 \sin(x), \quad y_2(0) = 0, \quad y_2'(0) = 0.9995$$

The frequency is $w = 1$ and the exact solutions are.

$$y_1(x) = \cos(x) + 0.0005x \sin(x)$$

$$y_2(x) = \sin(x) - 0.0005x \cos(x).$$

**Problem 3**: (Inhomogeneous linear system studied by Franco [29]).
\[
y'' + \frac{101}{2} y_1 - \frac{99}{2} y_2 = \frac{93}{2} \cos(2x) - \frac{99}{2} \sin(2x), \quad y_1(0) = 0, \quad y_1'(0) = -10.
\]
\[
y'' - \frac{99}{2} y_2 + \frac{101}{2} y_1 = \frac{93}{2} \sin(2x) - \frac{99}{2} \cos(2x), \quad y_1(0) = 1, \quad y_1'(0) = 12.
\]

The frequency is \( w = 10 \) and the exact solutions are.

\[
y_1(x) = -\cos(10x) - \sin(10x) + \cos(2x).
\]
\[
y_2(x) = \cos(10x) + \sin(10x) + \sin(2x).
\]

**Problem 4:** (Harmonic oscillator studied by Anastassi and Kosti [27]).

\[
y'' = -100y, \quad y(0) = 1, \quad y'(0) = 0.
\]

The frequency is \( w = 10 \) and the exact solution is \( y(x) = \cos(10x) \)

**Problem 5:** (Inhomogeneous equation studied by Al-Khasawneh *et al.* [30]).

\[
y''' = -y + x, \quad y(0) = 1, \quad y'(0) = 2.
\]

The frequency is \( w = 1 \) and the exact solution is \( y(x) = \sin(x) + \cos(x) + x \).

**Problem 6:** (Duffing equation [31]).

\[
y'''' = -y + y^2 + 0.002 \cos(1.01x), \quad y(0) = 0.200426728067, \quad y'(0) = 0.
\]

The frequency is \( w = 1 \), and the exact solution is.

\[
y(x) = 0.200179477536 \cos(1.01x) + 0.000246946143 \cos(3.03x)
+ 0.000000304014 \cos(5.05x) + 0.000000000374 \cos(7.07x).
\]

**CONCLUSION**

In the present paper, we have derived the new optimized Runge-Kutta-Nyström RKN method with vanishing the phase-lag, the first derivative of phase-lag and amplification error. Which is based on the Runge-Kutta-Nyström RKN method of sixth algebraic order, derived by Dormand, El-Mikkawy and Prince [24]. From the numerical results we conclude that, the new optimized RKN method is computationally more efficient and effective in solving special second order ODEs with oscillating solutions and outperformed the well-known RKN methods in the scientific literature.

**REFERENCES**


