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Exact Traveling Wave Solutions of Modified Liouville Equation

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Abstract: Generalized Bernoulli sub-ODE and exp(-Φ(ξ))-expansion methods are powerful tools for obtaining exact solutions of nonlinear partial differential equations. In this paper, these methods are applied to solve the modified Liouville equation. With the aid of mathematical software Maple13, some exact traveling wave solutions are established. When the parameters are taken as special values, the solitary wave solutions are originated from these traveling wave solutions. Further, three-dimensional plots of some of the solutions are given to visualize the dynamics of the equation. The results reveal that these two methods are very suitable and effective for solving nonlinear partial differential equations arising in mathematical physics.

Key words: Generalized Bernoulli Sub-ODE method · Exp(-Φ(ξ))-expansion method · Modified Liouville equation · Traveling wave solutions · Nonlinear partial differential equation

INTRODUCTION

Nonlinear phenomena are general problems in every field of engineering technology, science research, natural world and human society activities. So the investigation of exact solutions of nonlinear equations plays an important role not only in theoretic research but in application and they describe many different physical systems, ranging from gravitation to fluid dynamics. The interest of finding travelling wave solution of NPDEs is increasing and has now become a hot topic to researchers. In recent years, many researchers who are interested in the nonlinear physical phenomena have investigated exact solutions of NPDEs. With the development of soliton theory and the application of computer symbolic system such as Maple and Mathematica, many powerful methods for obtaining exact solutions of nonlinear evolution equations are presented, such as the tanh-method [1], the variational iteration method [2], the exp-function method [3], (G'/G)-expansion method [4], modified simple equation method [5] and so on.

Based on the previous works, we have studied the generalized sub-ODE [6] method and exp(-Φ(ξ))-expansion method [7] to construct exact traveling wave solutions for NPDEs through the modified Liouville equation.

The rest of the paper is organized as follows. In Section 2, we describe the Bernoulli sub-ODE method and the exp(-Φ(ξ))-expansion method. In Section 3, we will apply these methods to find exact traveling wave solutions of the modified Liouville equation. In Section 4, we sketch some graphs for various traveling wave solutions. In the last Section, some conclusions are presented.

MATERIALS AND METHODS

In this section, we will discuss the generalized sub-ODE method and exp(-Φ(ξ))-expansion method.

The generalized sub-ODE Method: Suppose that a nonlinear partial differential equation, say in two independent variables x and t, is given by

$$P(u,u_t,u_x,u_{tt},u_{xt},u_{xx},\cdots\cdots\cdots)=0, \tag{2.1}$$

where $u = u(x, t)$ is an unknown function, P is a polynomial in $u = u(x, t)$ and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved. In the following the main steps of the sub-ODE method are given:

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Step 1: The traveling wave variable $u(x, t) = u(\xi)$ where $\xi = x - ct$, permits us reducing Eq. (2.1) to an ODE for $u = u(\xi)$ in the form

$$P(u, -cu', u', c^2u'', -cu'', u'', \cdots\cdots\cdots) = 0, \tag{2.2}$$

Step 2: Suppose that the solution of Eq. (2.2) can be expressed by a polynomial in G as follows:

$$u(\xi) = \alpha_m H^m + \alpha_{m-1} H^{m-1} + \cdots\cdots\cdots \tag{2.3}$$

where $\alpha_m, \alpha_{m-1}, \ldots\ldots$ are constants to be determined later and $\alpha_m \neq 0$. The positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in Eq.(2.2) and $H = H(\xi)$ satisfies the following equation:

$$H' + \lambda H = \mu H^2 \tag{2.4}$$

When $\mu \neq 0$, Eq. (2.4) is the type of Bernoulli equation and we can obtain the solution as

$$H_1(\xi) = -\frac{\lambda}{2\mu}\left(\tanh\left(\frac{\lambda}{2}\xi\right) - 1\right) \tag{2.5}$$

$$H_2(\xi) = -\frac{\lambda}{2\mu}\left(\coth\left(\frac{\lambda}{2}\xi\right) - 1\right) \tag{2.6}$$

Step 3: Substituting Eq. (2.3) into Eq. (2.2) and using Eq. (2.4), collecting all terms with the same power of H together, the left-hand side of Eq. (2.3) is converted into another polynomial in H. Equating each coefficient of this polynomial to zero, yields a set of algebraic equations for $\alpha_m, \alpha_{m-1}, \ldots\ldots \lambda, \mu$.

Step 4: Solving the system of algebraic equations in Step 3 and by using the solutions of Eq. (2.4), we can construct the traveling wave solutions of the nonlinear evolution equation (2.1).

The exp(-Φ(ξ))-expansion method: We now present the *exp*(-*Φ*(ξ))-expansion method for solving the nonlinear evolution equation of the form of Eq. (2.1).

Step 1: Suppose the solution of Eq. (2.3) can be expressed in the following form

$$u(\xi) = \sum_{i=0}^{m} \alpha_i (\exp(-\Phi(\xi)))^i, \tag{2.7}$$

where α_i are constants, the positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and the nonlinear terms appearing in Eq. (2.2) and $\Phi = \Phi(\xi)$ satisfies the equation:

$$\Phi'(\xi) = \exp(-\Phi(\xi)) + \mu \exp(\Phi(\xi)) + \lambda \tag{2.8}$$

Eq. (2.8) gives the following solutions:

When $\lambda^2 - 4\mu > 0$, $\mu \neq 0$

$$\Phi_1(\xi) = \ln\left(\frac{-\sqrt{\lambda^2 - 4\mu}\tanh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}(\xi + E)\right) - \lambda}{2\mu}\right).$$

When $\lambda^2 - 4\mu < 0$, $\mu \neq 0$

$$\Phi_2(\xi) = \ln\left(\frac{\sqrt{4\mu - \lambda^2}\tan\left(\frac{\sqrt{4\mu - \lambda^2}}{2}(\xi + E)\right) - \lambda}{2\mu}\right)$$

When $\lambda^2 - 4\mu > 0$, $\mu = 0$, $\lambda \neq 0$

$$\Phi_3(\xi) = -\ln\left(\frac{\lambda}{\exp(\lambda(\xi + E)) - 1}\right)$$

When $\lambda^2 - 4\mu = 0$, $\mu \neq 0$, $\lambda \neq 0$

$$\Phi_4(\xi) = \ln\left(-\frac{2[\lambda(\xi + E) + 2]}{\lambda^2(\xi + E)}\right)$$

When $\lambda^2 - 4\mu = 0$, $\mu = 0$, $\lambda = 0$

$$\Phi_5(\xi) = \ln(\xi + E).$$

Step 3: We substitute Eq. (2.7) into Eq.(2.2) and use Eq.(2.1) and then we account the function exp(-Φ(ξ)). As a result of this substitution, we get a polynomial of exp(-Φ(ξ)). We equate all the coefficients of same power of exp(-Φ(ξ)) to zero. This procedure yields a system of algebraic equations whichever can be solved to find

$\alpha_m, \alpha_{m-1}, \ldots\ldots \lambda, \mu$ Substituting the values $\alpha_m, \alpha_{m-1}, \ldots\ldots \lambda, \mu$ into Eq. (2.7) along with general solutions of eq. (2.8) completes the determination of the solution of Eq. (2.1).

Application of the Methods for Modified Liouville Equation: In this section, we will study the generalized Bernoulli sub-ODE method and exp(-Φ(ξ))-expansion method to find the exact solutions of Liouville equation. Let us consider the Liouville equation [8, 9].

$$w_{tt} = a^2 w_{xx} + b e^{\beta w}, \tag{3.1}$$

that arises in hydrodynamics, where $w(x, t)$ is the stream function and a, b, β are nonzero constants.

We first use the Painleve transformation [10] $u(x,t) = e^{\beta w}$, so that

$$w = \frac{1}{\beta} \ln u. \tag{3.2}$$

Now the traveling wave transformation equations $u(x,t) = u(\xi)$, $\xi = x - ct$ and Eq.(3.2) reduces Eq.(3.1) into the following ODE:

$$uu'' - u'^2 + ku^3 = 0, \tag{3.3}$$

where $k = \frac{b\beta}{a^2 - c^2}$ and $c \neq \pm a$

Balancing the order of uu'' and u^3 in Eq.(3.3), we have $m = 0$.

Exact Solutions for Liouville Equation via Generalized Bernoulli sub-ODE Method: Through the generalized Bernoulli sub-ODE method, for $m=2$, Eq.(2.4) can be rewritten as follows:

$$u(\xi) = \alpha_2 H^2 + \alpha_1 H + \alpha_0, \ \ \alpha_2 \neq 0 \tag{3.4}$$

where $\alpha_2, \alpha_1, \alpha_0$ are constants to be determined later.

Substituting Eq.(3.4) into Eq.(3.3) and collecting all the terms with the same power of H together, equating each coefficient to zero, yields a set of simultaneous algebraic equations as follows:

$$\begin{aligned}
&3\alpha_0^3 = 0\\
&\alpha_0\alpha_1\lambda^2 + 3k\alpha_0^2\alpha_1 = 0\\
&\alpha_0(-3\alpha_1\mu\lambda + 4\alpha_2\lambda^2) + k(\alpha_0(2\alpha_0\alpha_2 + \alpha_1^2) + 2\alpha_1^2\alpha_0 + \alpha_2\alpha_0^2) = 0\\
&\alpha_0(-10\alpha_2\mu\lambda + 2\mu^2\alpha_1) + \alpha_1(-3\alpha_1\mu\lambda + 4\alpha_2\lambda^2) + \alpha_1\alpha_2\lambda^2 + 2\alpha_1\lambda(\alpha_1\mu - 2\alpha_2\lambda)\\
&+6\alpha_0\alpha_2\mu^2 + \alpha_1(-10\alpha_2\mu\lambda + 2\mu^2\alpha_1) + k(4\alpha_0\alpha_1\alpha_2 + \alpha_1(2\alpha_0\alpha_2 + \alpha_1^2)) = 0\\
&-4\alpha_1\alpha_2\lambda\mu - (\alpha_1\mu - 2\alpha_2\lambda)^2 + 6\alpha_0\alpha_2\mu^2 + \alpha_1(-10\alpha_2\mu\lambda + 2\mu^2\alpha_1)\\
&+\alpha_2(-3\alpha_1\mu\lambda + 4\alpha_2\lambda^2) + k(\alpha_0 + \alpha_2^2 + 2\alpha_1^2\alpha_2 + \alpha_2(2\alpha_0\alpha_2 + \alpha_1^2)) = 0\\
&-4(\alpha_1\mu - 2\alpha_2\lambda)\alpha_2\mu + 6\alpha_1\alpha_2\mu^2 + \alpha_2(-10\alpha_2\mu\lambda + 2\mu^2\alpha_1) + 3k\alpha_1\alpha_2^2 = 0\\
&2\alpha_2^2\mu^2 + k\alpha_2^3 = 0
\end{aligned}$$

Solving the algebraic equations above, yields:

$$\alpha_0 = 0, \alpha_1 = \frac{2\lambda\mu}{k}, \alpha_2 = \frac{-2\mu^2}{k} \ \text{ where } k \neq 0 \tag{3.5}$$

Substituting Eq. (3.5) into Eq. (3.4) along with Eq. (2.6) and Eq. (2.7), we obtain

$$u_1(\xi) = \frac{\lambda^2}{2k} \sec h^2\left(\frac{\lambda}{2}\xi\right). \tag{3.6}$$

$$u_1(\xi) = -\frac{\lambda^2}{2k} \csc h^2\left(\frac{\lambda}{2}\xi\right). \tag{3.7}$$

And finally using Eq. (3.2) the traveling wave solution of Eq. (3.1) becomes

$$w_1(\xi)=\frac{1}{\beta}\ln\left(\frac{\lambda^2}{2k}\sec h^2\left(\frac{\lambda}{2}\xi\right)\right), \tag{3.8}$$

$$w_2(\xi)=\frac{1}{\beta}\ln\left(-\frac{\lambda^2}{2k}\csc h^2\left(\frac{\lambda}{2}\xi\right)\right), \tag{3.9}$$

where $\xi = x - ct,\ k = \frac{b\beta}{a^2 - c^2}$ and $c \neq \pm a$.

Exact Solutions for Liouville Equation via exp(-Φ(ξ))-Expansion Method: Through the exp(-Φ(ξ))-expansion method, for *m=2,* Eq. (2.7) becomes

$$u(\xi)=\alpha_2(\exp(-\Phi(\xi)))^2+\alpha_1\exp(-\Phi(\xi))+\alpha_0,\ \ \alpha_2\neq 0 \tag{3.10}$$

where $\alpha_2,\alpha_1,\alpha_0$, are constants to be determined later.

Substituting eq.(3.10) into Eq.(3.3) and collecting all the terms with the same power of exp(−Φ(ξ)) together, equating each coefficient to zero, yields a set of simultaneous algebraic equations as follows:

$$2\alpha_2^2+k\alpha_2^3=0$$

$$3k\alpha_1\alpha_2^2+4\alpha_1\alpha_2+2\alpha_2^2\lambda=0$$

$$3k\alpha_1^2\alpha_2+6\alpha_0\alpha_2+\alpha_1^2+5\alpha_1\alpha_2\lambda+3k\alpha_0\alpha_2^2=0$$

$$\alpha_1^2\lambda+k\alpha_1^3+2\alpha_1\alpha_2\mu-2\alpha_2^2\mu\lambda+6k\alpha_0\alpha_1\alpha_2+\alpha_1\alpha_2\lambda^2+10\alpha_0\alpha_2\lambda+2\alpha_0\alpha_1=0$$

$$4\alpha_0\alpha_2\lambda^2+3k\alpha_0\alpha_1^2+3\alpha_0\alpha_1\lambda+3k\alpha_0^2\alpha_2-2\alpha_2^2\mu^2-\alpha_1\alpha_2\mu\lambda+8\alpha_0\alpha_2\mu=0$$

$$-\alpha_1^2\mu\lambda+3k\alpha_0^2\alpha_1+2\alpha_0\alpha_1\mu+6\alpha_0\alpha_2\mu\lambda-2\alpha_1\alpha_2\mu^2+\alpha_0\alpha_1\lambda^2=0$$

$$k\alpha_0^3+\alpha_0\alpha_1\lambda\mu+2\alpha_0\alpha_2\mu^2-\alpha_1^2\mu^2=0$$

Solving the algebraic equations above, yields:

$$\alpha_0=-\frac{2\mu}{k},\alpha_1=-\frac{2\lambda}{k},\alpha_2=-\frac{2}{k},k\neq 0. \tag{3.11}$$

Substituting Eq.(3.11) into Eq.(3.10), we get

$$u(\xi)=-\frac{2}{k}[\mu+\lambda\exp(-\Phi(\xi))+\exp(-2\Phi(\xi))] \tag{3.12}$$

And finally the exact solution of Eq. (3.1) is

$$w(\xi)=\frac{1}{\beta}\ln\left\{-\frac{2}{k}[\mu+\lambda\exp(-\Phi(\xi))+\exp(-2\Phi(\xi))]\right\} \tag{3.13}$$

When $\lambda^2-4\mu>0,\ \mu\neq 0$

$$w_1(\xi)=\frac{1}{\beta}\ln\left\{-\frac{2}{k}\left[\mu-\frac{2\mu\lambda}{\sqrt{\lambda^2-4\mu}\tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(\xi+E)\right)+\lambda}+\frac{4\mu^2}{\left\{\sqrt{\lambda^2-4\mu}\tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(\xi+E)\right)+\lambda\right\}^2}\right]\right\} \tag{3.14}$$

When $\lambda^2-4\mu<0$, $\mu\neq 0$

$$w_2(\xi)=\frac{1}{\beta}\ln\left\{-\frac{2}{k}\left[\mu+\frac{2\mu\lambda}{\sqrt{4\mu-\lambda^2}\tan\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(\xi+E)\right)-\lambda}+\frac{4\mu^2}{\left\{\sqrt{4\mu-\lambda^2}\tan\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(\xi+E)\right)-\lambda\right\}^2}\right]\right\} \tag{3.15}$$

When $\lambda^2-4\mu>0$, $\mu=0$, $\lambda\neq 0$

$$w_3(\xi)=\frac{1}{\beta}\ln\left\{-\frac{2}{k}\left[\mu+\frac{\lambda^2}{\exp(\lambda(\xi+E))-1}+\frac{\lambda^2}{\left[\exp(\lambda(\xi+E))-1\right]^2}\right]\right\} \tag{3.16}$$

When $\lambda^2-4\mu=0$, $\mu\neq 0$, $\lambda\neq 0$

$$w_4(\xi)=\frac{1}{\beta}\ln\left\{-\frac{2}{k}\left[\mu-\frac{\lambda^2(\xi+E)}{2\left[\lambda(\xi+E)+2\right]}+\frac{\lambda^4(\xi+E)^2}{\left[2\left(\lambda(\xi+E)+2\right)\right]^2}\right]\right\} \tag{3.17}$$

When $\lambda^2-4\mu=0$, $\mu=0$, $\lambda=0$

$$w_5(\xi)=\frac{1}{\beta}\ln\left\{-\frac{2}{k}\left[\mu-\frac{\lambda}{\xi+E}+\frac{1}{(\xi+E)^2}\right]\right\} \tag{3.18}$$

where $\xi=x-ct,\ k=\frac{b\beta}{a^2-c^2}$ and $c\neq\pm a$.

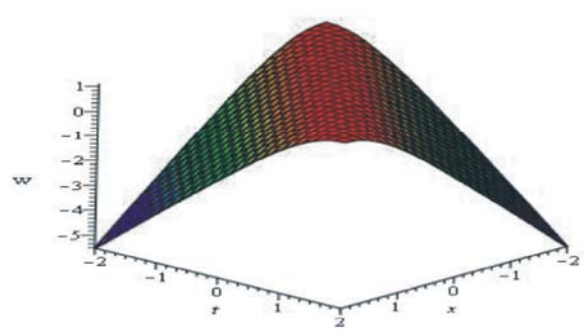

Fig. 1: Profile of Eq. (3.8) for a=2, b=2, β=1, λ=1, c=1 within the interval $-2\leq x,t\leq 2$.

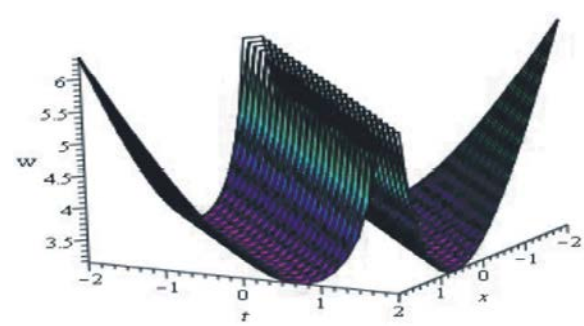

Fig. 2: Singular soliton profile of Eq. (3.9) for a=2, b=2, β=1, λ=1, c=1 within the interval $-2\leq x,t\leq 2$.

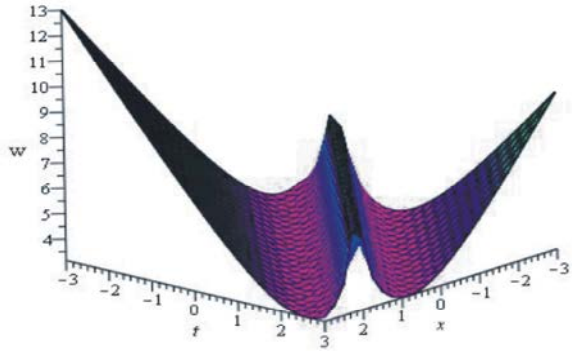

Fig. 3: Soliton profile of Eq.(3.14) for a=2, b=2, β=1, λ=3, μ=1, E=0,c=1 within the interval $-3 \le x,t \le 3$.

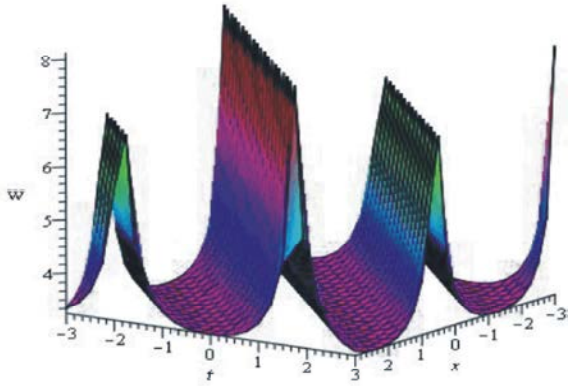

Fig. 4: Periodic profile of Eq.(3.15) for a=2, b=2, β=1, λ=1,μ=1, E=0,c=1 within the interval $-3 \le x,t \le 3$.

Remark: All the obtained results has been checked with Maple by putting them back into the original equation and found correct.

Graphical Representation: The graphical demonstrations of some obtained solutions of Liouville equation are shown in Fig. 1-Fig. 4.

CONCLUSIONS

In this article, generalized sub-ODE method and exp(-Φ(ξ))-expansion method have been successfully implemented to find exact traveling wave solutions of Liouville equation. We obtain some new traveling wave solutions including hyperbolic function solutions, trigonometric function solutions and rational solutions. These results show that the two methods are reliable and effective and can be also applied to other kinds of NPDEs.

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