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Numerical Solutions of Cubic and Modified Cubic Boussinesq Equation Using Homotopy Analysis Method

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Abstract: Nonlinear problems occur in different fields of sciences and engineering. Most of the problems can be modeled by linear and nonlinear partial differential equations. It is very difficult and tedious to find the exact solutions of nonlinear models. The most important and challenging task is to find the exact solutions of such models. Therefore many researchers prefer to investigate the approximate solutions of such problems. In this study, Cubic Boussinesq equation (CBE) and Modified Cubic Boussinesq equation (MCBE) has been solved using a well-known iterative technique called the homotopy analysis method (HAM). Results obtained by HAM are compared with the exact solutions and it is shown that, the results obtained by HAM have a great agreement with the exact solution. Numerical results are elaborated with the help of absolute error tables.

Key words: Homotopy Analysis Method · Cubic Boussinesq Equation · Modified Cubic Boussinesq Equation · Approximate Solution

INTRODUCTION

In 1992, Liao developed a technique for the solution of nonlinear problems, namely homotopy analysis method [1, 2]. He applied this method to solve different kinds of differential equations [3, 4]. The method is also effectively applied by various researchers to solve wide class of partial differential equations [5-9]. All these applications verified the validity, effectiveness and flexibility of the HAM. The advantage of HAM over other methods is that it contains an auxiliary parameter '$\hbar$', which can be simply used to adjust and control the convergence region of series solution.

To illustrate the basic idea of HAM, we consider the following nonlinear differential equation.

$$N[u(x, t)] = 0, \tag{1}$$

where N is supposed to be nonlinear operator, $u(x, t)$ is an unidentified function. For the sack of ease, all the boundary and initial conditions have been ignored. Liao created the following zero order deformation equation using the conventional homotopy.

$$(1-p)L\left[\phi(x,t;p)-u_0(x,t)\right]=p\hbar H(x,t)N\left[\phi(x,t;p)\right]=0 \tag{2}$$

where $L=\frac{d}{dt}$, $p \in [0,1]$ is the embedding parameter, $\hbar \neq 0$ is a non-zero auxiliary parameter,
$H(x, t) \neq 0$ is an auxiliary function, L is linear operator, $\phi(x; t; p)$ is an unknown function and $u_0(x, t)$ is the initial guess of $u(x, t)$ When $p = 0$ and $p = 1$, equation (2) gives,

$$\phi(x; t; 0) = u_0 (x, t), \quad \phi(x; t; 1) = u(x, t) \tag{3}$$

Now expanding $\phi(x; t; p)$ with respect to p by Taylor series we have,

$$\phi(x,t;p)=u_0(x,t)+\sum_{n=1}^{+\infty}u_n(x,t)p^n \tag{4}$$

where,

$$u_n(x,t)=\frac{1}{n!}\frac{\partial^n\phi(x,t;p)}{\partial p^n}\bigg|_{p=0}. \tag{5}$$

If we choose the linear operator L, the initial guess $u_0(x, t)$ the parameter $\hbar$ and the auxiliary function $H(x, t)$ properly, the series (4) converges at $p = 1$, then we get,

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$$u(x,t)=\phi(x,t;1)=u_0(x,t)+\sum_{n=1}^{+\infty}u_n(x,t). \tag{6}$$

Define the vector,

$$\vec{u}(x,t)=\{u_0(x,t),u_1(x,t),u_2(x,t),...,u_n(x,t)\}. \tag{7}$$

To obtain the n[th] order deformation equation, differentiate equation (2) with respect to p and then setting $p = 0$ and finally dividing by $n!$, we get

$$L[u_n-\lambda_n u_{n-1}]=\hbar H(x,t)R_n\left[\vec{u}_{n-1}\right]. \tag{8}$$

where,

$$R_n\left[\vec{u}_{n-1}\right]=\frac{1}{(n-1)!}\frac{\partial^{n-1}N\left[\phi(x,t;p)\right]}{\partial u^{n-1}} \tag{9}$$

and

$$\lambda_n=\begin{cases}0, & n=1\\ 1, & n>1\end{cases}. \tag{10}$$

Now applying L^{-1} on both sides of equation (8), we get

$$\left[u_n-\lambda_n u_{n-1}\right]=\hbar L^{-1}\left\{H(x,t)R_n\left[\vec{u}_{n-1}\right]\right\}, \tag{11}$$

$$u_n(x,t)=\hbar L^{-1}\left\{H(x,t)R_n\left[\vec{u}_{n-1}\right]\right\}+\lambda_n u_{n-1}. \tag{12}$$

In this way we can easily obtain the approximate solution $u(x, t)$

$$u(x,t)=\sum_{n=0}^{N}u_n(x,t). \tag{13}$$

where $u_n(x, t)$ can be obtain from equation (12).

Numerical Examples: In the following section we will provide numerical solution of cubic Boussinesq equation and cubic modified Boussinesq equation.

Example 1: Consider the cubic Boussinesq equation.

$$u_{tt}-u_{xx}-u_{xxxx}+2\left(u^3\right)_{xx}=0 \tag{14}$$

Here the initial condition is,

$$u(x,0)=\frac{1}{x}$$

and the exact solution of equation (14) as given in [10] is,

$$u(x,t)=\frac{1}{x+t} \tag{15}$$

Using equation (14) in equation (12), we get

$$u_n(x,t)=hL^{-1}\left\{H(x,t)R_n\left[\vec{u}_{n-1}\right]\right\}+\lambda_n u_{n-1}. \tag{16}$$

where,

$$\Re_n\left(\vec{u}_{n-1}\right)=\frac{\partial^2 u_{n-1}}{\partial t^2}-\frac{\partial^2 u_{n-1}}{\partial x^2}-\frac{\partial^4 u_{n-1}}{\partial x^4}+12\sum_{i=0}^{n-1}u_i\frac{\partial u_i}{\partial x}\frac{\partial u_{n-1-i}}{\partial x}+6\sum_{i=0}^{n-1}u_i^2\frac{\partial^2 u_{n-1-i}}{\partial x^2} \tag{17}$$

Here we assume $H(x, t) = 1$. For $n = 1$, equation (16) and equation (17) gives;

$$u_1(x,t)=\hbar L^{-1}\left\{R_1\left[\vec{u}_0\right]\right\}+\lambda_1 u_0. \tag{18}$$

and

$$\Re_1\left(\vec{u}_0\right)=\frac{\partial^2 u_0}{\partial t^2}-\frac{\partial^2 u_0}{\partial x^2}-\frac{\partial^4 u_0}{\partial x^4}+12u_0\frac{\partial u_0}{\partial x}\frac{\partial u_0}{\partial x}+6u_0^2\frac{\partial^2 u_0}{\partial x^2} \tag{19}$$

Using equation (19) in equation (18), we get the first approximation,

$$u_1(x,t)=\frac{-2ht}{x^3}$$

Similarly one can find u_2, u_3, u_4,...

$$u_2(x,t)=h\left(\frac{12ht^2}{x^5}+\frac{252ht^2}{x^7}+\frac{64h^2t^2}{x^9}\right)-\frac{2ht}{x^3}$$

$$u_3(x,t)=-\frac{2ht}{35x^{21}}\begin{pmatrix}-36288h^4t^4x^8-420htx^{16}+44100h^3t^3x^{10}-21337344h^4t^4x^4\\ -1702848h^4t^4x^6-10160640h^5t^5x^2+3462480h^3t^3x^8-430080h^5t^5x^4\\ -8820htx^{14}-3360h^4x^{12}t+2100h^2t^2x^{14}+180880h^2t^2x^{12}\\ +6791400h^2t^2x^{10}-420h^2x^{16}-8820h^2x^{14}-1228800h^6t^6+35x^{18}\end{pmatrix}$$

Substituting u_0, u_1, u_2, u_3, u_4,... in equation (6) we have

$$u(x,t)=\frac{-2ht}{x^3}+h\left(\frac{12ht^2}{x^5}+\frac{252ht^2}{x^7}+\frac{64h^2t^2}{x^9}\right)-\frac{2ht}{x^3}$$
$$-\frac{2ht}{35x^{21}}\begin{pmatrix}-36288h^4t^4x^8-420htx^{16}+44100h^3t^3x^{10}-21337344h^4t^4x^4\\-1702848h^4t^4x^6-10160640h^5t^5x^2+3462480h^3t^3x^8-430080h^5t^5x^4\\-8820htx^{14}-3360h^4x^{12}t+2100h^2t^2x^{14}+180880h^2t^2x^{12}\\+6791400h^2t^2x^{10}-420h^2x^{16}-8820h^2x^{14}-1228800h^6t^6+35x^{18}\end{pmatrix}\dots \quad (20)$$

Example 2: Consider the cubic modified Boussinesq equation,

$$u_{tt}+u_{xxt}+\frac{2}{9}u_{xxxx}-\left(u^3\right)_{xx}=0 \quad (21)$$

Here the initial condition is

$$u(x,0)=1+\tanh\left(\frac{3}{2}x\right) \quad (22)$$

and the exact solution is given by [11]

$$u(x,y)=1+\tanh\left(\frac{3}{2}x-3t\right) \quad (23)$$

We have,

$$\Re_m\left(\vec{u}_{m-1}\right)=\frac{\partial^2u_{m-1}}{\partial t^2}+\frac{\partial}{\partial t}\left(\frac{\partial^2u_{m-1}}{\partial x^2}\right)+\frac{2}{9}\frac{\partial^4u_{m-1}}{\partial x^4}-6\sum_{i=0}^{m-1}u_i\frac{\partial u_i}{\partial x}\frac{\partial u_{m-1-i}}{\partial x}-3\sum_{i=0}^{m-1}u_i{}^2\frac{\partial^2u_{m-1-i}}{\partial x^2}$$

We get

$$u_1(x,t)=-\frac{9ht}{2}\left(\tanh\left(\frac{3}{2}x\right)^2-1\right)\left(4\tanh\left(\frac{3}{2}x\right)-3+9\tanh\left(\frac{3}{2}x\right)^2\right)$$

$$\begin{aligned}u_2(x,t)&=810h^2t\tanh\left(\frac{3}{2}x\right)^3-\frac{371061}{8}h^3t^3\tanh\left(\frac{3}{2}x\right)^3+\frac{562059}{4}h^3t^3\tanh\left(\frac{3}{2}x\right)^5\\
&+\frac{8019}{2}h^2t^2\tanh\left(\frac{3}{2}x\right)^7-486h^2t\tanh\left(\frac{3}{2}x\right)^5+18ht\tanh\left(\frac{3}{2}x\right)\\
&-\frac{196101}{4}h^2t^2\tanh\left(\frac{3}{2}x\right)^4-\frac{41553}{2}h^3t^3\tanh\left(\frac{3}{2}x\right)-\frac{809919}{4}h^3t^3\tanh\left(\frac{3}{2}x\right)^7\\
&-\frac{59049}{2}h^3t^3\tanh\left(\frac{3}{2}x\right)^{10}-142155h^3t^3\tanh\left(\frac{3}{2}x\right)^6-\frac{4131}{2}h^2t\tanh\left(\frac{3}{2}x\right)^2\\
&\frac{188811}{4}h^2t^2\tanh\left(\frac{3}{2}x\right)^6-\frac{81}{2}ht\tanh\left(\frac{3}{2}x\right)^4+\frac{74601}{4}h^2t^2\tanh\left(\frac{3}{2}x\right)^2\\
&-\frac{13365}{2}h^2t^2\tanh\left(\frac{3}{2}x\right)^5-324h^2t\tanh\left(\frac{3}{2}x\right)+\frac{567}{2}h^2t^2\tanh\left(\frac{3}{2}x\right)\\
&+54ht\tanh\left(\frac{3}{2}x\right)^2+\frac{1114641}{8}h^3t^3\tanh\left(\frac{3}{2}x\right)^9-\frac{3645}{2}h^2t\tanh\left(\frac{3}{2}x\right)^6\\
&\frac{4779}{2}h^2t^2\tanh\left(\frac{3}{2}x\right)^3+\frac{47385}{8}h^3t^3\tanh\left(\frac{3}{2}x\right)-18ht\tanh\left(\frac{3}{2}x\right)^3\\
&-15309h^2t^2\tanh\left(\frac{3}{2}x\right)^8-\frac{295245}{8}h^3t^3\tanh\left(\frac{3}{2}x\right)^{11}+\frac{212139}{2}h^3t^3\tanh\left(\frac{3}{2}x\right)^8\\
&+85293h^3t^3\tanh\left(\frac{3}{2}x\right)^4+3645h^2t\tanh\left(\frac{3}{2}x\right)^4-\frac{6075}{4}h^2t^2+243h^2t+\frac{2187}{2}h^3t^3-\frac{27}{2}ht\end{aligned}$$

Approximate solution can be calculated from equation (13)

$$u(x,t)=u_0+u_1+u_2+\dots$$

Putting values in above equation we get the series solution as,

$$
\begin{aligned}
u(x,t) = {} & 1+\tanh\left(\frac{3}{2}x\right)-\frac{9ht}{2}\left(\tanh\left(\frac{3}{2}x\right)^2-1\right)\left(4\tanh\left(\frac{3}{2}x\right)-3+9\tanh\left(\frac{3}{2}x\right)^2\right) \\
& 810h^2t\tanh\left(\frac{3}{2}x\right)^3-\frac{371061}{8}h^3t^3\tanh\left(\frac{3}{2}x\right)^3+\frac{562059}{4}h^3t^3\tanh\left(\frac{3}{2}x\right)^5 \\
& +\frac{8019}{2}h^2t^2\tanh\left(\frac{3}{2}x\right)^7-486h^2t\tanh\left(\frac{3}{2}x\right)^5+18ht\tanh\left(\frac{3}{2}x\right) \\
& -\frac{196101}{4}h^2t^2\tanh\left(\frac{3}{2}x\right)^4-\frac{41553}{2}h^3t^3\tanh\left(\frac{3}{2}x\right)-\frac{809919}{4}h^3t^3\tanh\left(\frac{3}{2}x\right)^7 \\
& -\frac{59049}{2}h^3t^3\tanh\left(\frac{3}{2}x\right)^{10}-142155h^3t^3\tanh\left(\frac{3}{2}x\right)^6-\frac{4131}{2}h^2t\tanh\left(\frac{3}{2}x\right)^2 \\
& \frac{188811}{4}h^2t^2\tanh\left(\frac{3}{2}x\right)^6-\frac{81}{2}ht\tanh\left(\frac{3}{2}x\right)^4+\frac{74601}{4}h^2t^2\tanh\left(\frac{3}{2}x\right)^2 \\
& -\frac{13365}{2}h^2t^2\tanh\left(\frac{3}{2}x\right)^5-324h^2t\tanh\left(\frac{3}{2}x\right)+\frac{567}{2}h^2t^2\tanh\left(\frac{3}{2}x\right) \\
& +54ht\tanh\left(\frac{3}{2}x\right)^2+\frac{1114641}{8}h^3t^3\tanh\left(\frac{3}{2}x\right)^9-\frac{3645}{2}h^2t\tanh\left(\frac{3}{2}x\right)^6 \\
& \frac{4779}{2}h^2t^2\tanh\left(\frac{3}{2}x\right)^3+\frac{47385}{8}h^3t^3\tanh\left(\frac{3}{2}x\right)-18ht\tanh\left(\frac{3}{2}x\right)^3 \\
& -15309h^2t^2\tanh\left(\frac{3}{2}x\right)^8-\frac{295245}{8}h^3t^3\tanh\left(\frac{3}{2}x\right)^{11}+\frac{212139}{2}h^3t^3\tanh\left(\frac{3}{2}x\right)^8 \\
& +85293h^3t^3\tanh\left(\frac{3}{2}x\right)^4+3645h^2t\tanh\left(\frac{3}{2}x\right)^4-\frac{6075}{4}h^2t^2+243h^2t+\frac{2187}{2}h^3t^3-\frac{27}{2}ht \qquad (24)
\end{aligned}
$$

Table 1:

t_i / x_i	10	15	20	25	30
0.01	4.296×10^{-5}	2.703×10^{-5}	1.759×10^{-5}	1.219×10^{-5}	8.902×10^{-6}
0.02	8.603×10^{-5}	5.412×10^{-5}	3.520×10^{-5}	2.439×10^{-5}	1.781×10^{-5}
0.03	1.292×10^{-4}	8.125×10^{-5}	5.283×10^{-5}	3.661×10^{-5}	2.672×10^{-5}
0.04	1.725×10^{-4}	1.084×10^{-4}	7.049×10^{-5}	4.883×10^{-5}	3.565×10^{-5}
0.05	2.159×10^{-4}	1.356×10^{-4}	8.817×10^{-5}	6.107×10^{-5}	4.458×10^{-5}

Error Analysis: Absolute error between exact solution (15) and approximate solution (20) up to third order iteration for h = -1.

Table 2:

t_i / x_i	10	15	20	25	30
0.01	5.000×10^{-9}	$1.010^{\circ}10^{-8}$	0.000	5.000×10^{-9}	0.000
0.02	0.000	0.000	0.000	0.000	0.000
0.03	1.010×10^{-8}	0.000	1.010×10^{-8}	0.000	0.000
0.04	0.000	0.000	0.000	0.000	0.000
0.05	1.100×10^{-8}	0.000	5.000×10^{-8}	0.000	0.000

Error Analysis: Absolute error between exact solution (23) and approximate solution (24) up to second order iteration.

CONCLUSION

In this work, the HAM is used to get the approximate solution of Cubic and Modified Cubic Boussinesq equations using Homotopy Analysis Method. Maple has been used to generate analytical results. The numerical results in Table (1) and Table (2) show that, the approximate solution has a great agreement with the exact solution. Using the error tables one can conclude that this method is a powerful tool to find the approximate (some time exact) solutions for many problems. It is worth mentioning here that this method gives rapid convergence of series solution due to its auxiliary parameter.

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