Corresponding Author: M. Aslam Malik, Department of Mathematics, University of the Punjab, Quaid-e-Azam Campus, Lahore-54590, Pakistan.

2123

Abstract: The notion of soft BCI-commutative ideals and BCI-commutative idealistic soft BCI-algebras is introduced and their basic properties are discussed. Relations between soft ideals and soft BCI-commutative ideals of soft BCI-algebras are provided. Also idealistic soft BCI-algebras and BCI-commutative idealistic soft BCI-algebras are being related. The intersection, union, “AND” operation and “OR” operation of soft BCI-commutative ideals and BCI-commutative idealistic soft BCI-algebras are established. The characterizations of (fuzzy) BCI-commutative ideals in BCI-algebras are given by using the concept of soft sets. Relations between fuzzy BCI-commutative ideals and BCI-commutative idealistic soft BCI-algebras are discussed.

Key words: Soft set - (BCI-commutative idealistic) soft BCI-algebra - Soft ideal - Soft BCI-commutative ideal

INTRODUCTION

Qin and Hong [2] discussed the algebraic structure of soft sets and constructed the lattice structures of soft sets. They also introduced the concept of soft equality and derived some related properties. Cagman et al. [3] extended soft sets to fuzzy parameterized soft sets and discussed their applications in decision making problem. Yang et al. [4] introduced the concept of the interval valued soft set by combing the interval-valued fuzzy set and soft set models. They defined complement, “AND” and “OR” operations on the interval-valued soft sets. They also proved De Morgan’s, associative and distributive laws of the interval-valued fuzzy soft sets. Gong et al. [5] proposed the concept of bijective soft set and some of its operations such as the restricted AND and the relaxed AND operations. They also discussed dependency between two bijective soft sets, bijection soft decision system, significance of bijective soft set with respect to bijective soft set decision system, reduction of bijective soft set with respect to bijective soft decision system and decision rules in bijective soft decision system. Aktas and Cagman [6] discussed the basic properties of soft sets and compared soft sets to the related concepts of fuzzy sets and rough sets. They also introduced soft groups and derived their basic properties. Aygunoglu and Aygun [7] introduced the concept of fuzzy soft groups and discussed their structural
characteristics. They also defined fuzzy soft function and fuzzy soft homomorphism and gave theorems of homomorphic image and homomorphic pre-image. Feng et al. [8] introduced the notions of soft semi-rings, soft sub semi-rings, idealistic soft semi-rings and soft semi-ring homomorphisms and investigated related properties. Acar et al. [9] defined soft rings and introduced their basic properties such as soft ideals, soft homomorphisms etc. by using soft set theory. We refer the readers to [10, 11] for further information regarding development of soft set theory.

Jun [12] applied the concept of soft sets by Molodtso to the theory of BCK/BCI-algebras. He introduced the notion of soft BCK/BCI-algebras and soft subalgebras. Jun et al. [13] introduced the notion of soft p-ideals and p-idealistic soft BCI-algebras and provided the relations between fuzzy p-ideals and p-idealistic soft BCI-algebras. In [14] Jun et al. further introduced the notions of fuzzy soft BCK/BCI-algebras, (closed) fuzzy soft ideals and fuzzy soft p-ideals and discussed the related properties. In this paper, we introduce the notion of soft BCI-commutative ideals and BCI-commutative idealistic soft BCI-algebras. Using soft sets, we give characterizations of (fuzzy) BCI-commutative ideals in BCI-algebras. We provide relations between fuzzy BCI-commutative ideals and BCI-commutative idealistic soft BCI-algebras.

**Basic Results on BCI-Algebras:** BCK/BCI-algebras are important classes of logical algebras introduced by Y. Imai and K. Iseki [15] and were extensively investigated by several researchers.

An algebra \((X,*,0)\) of type \((2,0)\) is called an abCI-algebra if it satisfies the following conditions:

(i) \(((x*y)*(x*z))*(z*y) = 0\),  
(ii) \((x*y)*y = 0\)

for all \(x, y, z \in X\). In a BCI-algebra \(X\), we can define a partial ordering \(\leq\) by putting \(x \leq y\) if and only if \(x*y = 0\). If a BCI-algebra \(X\) satisfies the identity:

(iii) \(0*x = 0\),

for all \(x \in X\), then \(X\) is called a BCK-algebra.

In any BCI-algebra the following hold:

(iv) \((x*y)*z = (x*z)*y\),  
(vi) \(x*0 = x\)

(vii) \(x \leq y\) implies \(x*z \leq y*z\) and \(z*y \leq z*x\)

(ix) \(0*(x*y) = 0*(0*x)*(x*y)\), \((x) x*(x*y*y) = (x*y)\)

(xi) \((x*z)*(y*z) \leq x*y\)

for all \(x, y, z \in X\).

A non-empty subset \(S\) of a BCI-algebra \(X\) is called a sub-algebra of \(X\) if \(x*y \in S\) for all \(x, y \in S\). A non-empty subset \(I\) of a BCI-algebra \(X\) is called an ideal of \(X\) if for any \(x \in X\),

(i) \(0 \in I\)

(ii) \(x*y \in I\) and \(y \in I\) implies \(x \in I\)

Any ideal \(I\) of a BCI-algebra \(X\) satisfies the following implication:

\(x \leq y\) and \(y \in I \Rightarrow x \in I\), \(\forall x \in X\)

A non-empty subset \(I\) of a BCI-algebra \(X\) is called an BCI-commutative ideal (see Meng [16]) of \(X\) if it satisfies

(I) and

\((x*y)*z \in I\) and \(z \in I \Rightarrow x*(y*(z*x))*(0*(0*z*y))) \in I\)

for all \(x, y \in X\). We know that every BCI-commutative ideal of a BCI-algebra \(X\) is also an ideal of \(X\).

We refer the readers to [17, 18] for further study about ideals in BCK/BCI-algebras.

**Basic Results on Soft Sets:** In [1] the soft set is defined in the following way: Let \(U\) be an initial universe set and \(E\) be a set of parameters. Let \(\mathcal{P}(U)\) denotes the power set of \(U\) and \(A \subset E\).

**Definition 3.1 (Molodtsov [1]):** A pair \((F, A)\) is called a soft set over \(U\), where \(F\) is a mapping given by \(F : A \rightarrow \mathcal{P}(U)\).

In other words, a soft set over \(U\) is a parameterized family of subsets of the universe \(U\). For \(a \in A\), \(F(a)\) may be considered as the set of \(a\)-approximate elements of the soft set \((F, A)\).

**Definition 3.2 (Maji et al. [19]):** Let \((F, A)\) and \((G, B)\) be two soft sets over a common universe \(U\). The intersection of \((F, A)\) and \((G, B)\) is defined to be the soft set \((H, C)\) satisfying the following conditions:

(i) \(C = A \cap B\)
(ii) \( H(x) = F(x) \) or \( G(x) \) for all \( x \in C \), (as both are same sets)

In this case, we write \( (F, A) \land (G, B) = (H, C) \).

**Definition 3.3 (Maji et al. [19]):** Let \( (F, A) \) and \( (G, B) \) be two soft sets over a common universe \( U \). The union of \( (F, A) \) and \( (G, B) \) is defined to be the soft set \((H, C)\) satisfying the following conditions:

(i) \( C = A \cup B \)

(ii) for all \( x \in C \), 
\[ \gamma(x) = \begin{cases} F(x) & \text{if } x \in A \setminus B \\ G(x) & \text{if } x \in B \setminus A \\ F(x) \cup G(x) & \text{if } x \in A \cap B \end{cases} \]

In this case, we write \( (F, A) \lor (G, B) = (H, C) \).

**Definition 3.4 (Maji et al. [19]):** Let \( (F, A) \) and \( (G, B) \) be two soft sets over a common universe \( U \). Then \( ((F, A) \land (G, B)) \) denoted by \( (F, A) \land (G, B) \) is defined as \( (F, A) \land (G, B) = (H, A \times B) \), where \( \gamma(x, y) = F(x) \cap G(y) \) for all \( (x, y) \in A \times B \).

**Definition 3.5 (Maji et al. [19]):** Let \( (F, A) \) and \( (G, B) \) be two soft sets over a common universe \( U \). Then \( ((F, A) \lor (G, B)) \) denoted by \( (F, A) \lor (G, B) \) is defined as \( (F, A) \lor (G, B) = (H, A \times B) \), where \( H(x, y) = F(x) \cup G(y) \) for all \( (x, y) \in A \times B \).

**Definition 3.6 (Maji et al. [19]):** For two soft sets \( (F, A) \) and \( (G, B) \) over a common universe \( U \), we say that \( (F, A) \) is a soft subset of \( (G, B) \), denoted by \( (F, A) \subseteq (G, B) \), if it satisfies:

(i) \( A \subseteq B \)

(ii) For every \( a \in A \), \( F(a) \) and \( G(a) \) are identical approximations.

**Soft BCI-Commutative Ideals:** In what follows let \( X \) and \( A \) be a BCI-algebra and a nonempty set, respectively and \( R \) will refer to an arbitrary binary relation between an element of \( A \) and an element of \( X \), that is, \( R \) is a subset of \( A \times X \) without otherwise specified. A set valued function \( F: A \rightarrow \mathcal{P}(X) \) can be defined as \( F(x) = \{ y \in X \mid x R y \} \) for all \( x \in A \). The pair \( (F, A) \) is then a soft set over \( x \).

**Definition 4.1 (Jun and Park [20]):** Let \( S \) be a sub-algebra of \( X \). A subset \( I \) of \( X \) is called an ideal of \( X \) related to \( S \) (briefly, \( S \)-ideal of \( X \)), denoted by \( I \triangleleft S \), if it satisfies:

(i) \( 0 \in I \)

(ii) \( x \ast y \in I \) and \( y \in I \Rightarrow x \in I \) for all \( x \in S \)

**Definition 4.2:** Let \( S \) be a subalgebra of \( X \). A subset \( I \) of \( X \) is called a BCI-commutative ideal of \( X \) related to \( S \) (briefly, \( S \)-(BCI–C)–ideal of \( X \)), denoted by \( I \triangleleft_{BCI-C} S \), if it satisfies:

(i) \( 0 \in I \)

(ii) \( (x \ast y) \ast z \in I \) and \( z \in I \Rightarrow x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))) \in I \) for all \( x, y, z \in S \)

**Example 4.3:** Let \( X = \{0, a, b, c\} \) be a BCI-algebra, with the following Cayley table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>a</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>c</td>
<td>b</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>c</td>
<td>0</td>
<td>a</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>b</td>
<td>a</td>
<td>0</td>
</tr>
</tbody>
</table>

Then \( S = \{0, b\} \) is a subalgebra of \( X \) and \( I = \{0, a, b\} \) is an \( S \)-(BCI–C) ideal of \( X \).

Note that every \( S \)-(BCI–C) ideal of \( X \) is an \( S \)-ideal of \( X \).

**Definition 4.4 (Jun [12]):** Let \( (F, A) \) be a soft set over \( X \). Then \( (F, A) \) is called a soft BCI-algebra over \( X \) if \( F(x) \) is a sub-algebra of \( X \) for all \( x \in A \).

**Definition 4.5 (Jun and Park [20]):** Let \( (F, A) \) be a soft BCI-algebra over \( X \). A soft set \( (G, I) \) over \( X \) is called a soft ideal of \( (F, A) \), denoted \( (G, I) \triangleleft (F, A) \), if it satisfies:

(i) \( I \subseteq A \)

(ii) \( G(x) \triangleleft F(x) \) for all \( x \in I \).

**Definition 4.6:** Let \( (F, A) \) be a soft BCI-algebra over \( X \). A soft set \( (G, I) \) over \( IX \) is called a soft BCI-commutative ideal of \( (F, A) \), denoted \( (G, I) \triangleleft_{BCI-C} (F, A) \), if it satisfies:

(i) \( I \subseteq A \)

(ii) \( G(x) \triangleleft_{BCI-C} F(x) \) for all \( x \in I \).

Let us illustrate this definition using the following example.

**Example 4.7:** Consider a BCI-algebra \( X = \{0, a, b, c\} \) which is given in Example 4.3. Let \( (F, A) \) be a soft set over \( X \), where \( A = X \) and \( F: A \rightarrow \mathcal{P}(X) \) is a set-valued function defined by: \( F(x) = \{0\} \cup \{y \in X \mid x \ast y \in \{0, a\}\} \) for all \( x \in A \).
Then \( F(0) = F(a) = X, F(b) = F(c) = \emptyset \), which are subalgebras of \( X \). Hence \((F, A)\) is a soft BCI-algebra over \( X \). Let \( I = \{0,a,b\} \subset A \) and \( G : I \to \Psi(X) \) be a set-valued function defined by:

\[
G(x) = \begin{cases} 
Z(\{0,a\}) & \text{if } x = b \\
\emptyset & \text{if } x \in \{0,a\}
\end{cases}
\]

where \( Z(\{0,a\}) = \{x \in X | 0 \leq (0, x) \in [0,a]\} \).

Then \( G(0) = \emptyset \) since \( b \leq c \).
X = F(a), G(b) = [0,a] and \( G(c) = [0,a] \) are subalgebras of \( X \). Hence \((F, A)\) is a soft BCI-algebra over \( X \). Let \((G, I)\) be a soft set over \( X \) where \( I = \{a,b\} \subset A \) and \( G : I \to \Psi(X) \) be a set-valued function defined by \( G(x) = [y \in X | y \preceq x = 0] \) for all \( x \in I \). Then \( G(a) = [0,a] \preceq X = F(a), G(b) = [0,a] \preceq [0,a,c,d] = F(b) \). Hence \((G, I)\) is a soft ideal of \((F, A)\) but it is not a soft BCI-commutative ideal of \((F, A)\) because \( G(a) \) is not a BCI-commutative ideal of \( X \) related to \( F(a) \) since \((b \preceq c) \preceq a = 0 \in G(a) \) and \( a \in G(a) \) but \( b \preceq (c \preceq c \preceq b) \preceq 0 \in G(a) \).

**Example 4.8:** Let \( X = \{0,a,b,c,d\} \) be a BCK-algebra and hence a BCI-algebra, with the following Cayley table:

<table>
<thead>
<tr>
<th>( \ast )</th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>d</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>b</td>
<td>0</td>
<td>0</td>
<td>d</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>c</td>
<td>c</td>
<td>0</td>
<td>d</td>
</tr>
<tr>
<td>d</td>
<td>d</td>
<td>d</td>
<td>d</td>
<td>d</td>
<td>d</td>
</tr>
</tbody>
</table>

Let \((F, A)\) be a soft set over \( X \), where \( A = X \) and \( F : A \to \Psi(X) \) is a set-valued function defined by \( F(x) = \emptyset \cup \{y \in X | y \ast x \preceq 0\} \) for all \( x \in A \). Then \( F(0) = F(a) = X, F(b) = \emptyset \) and \( F(c) = \emptyset \), which are subalgebras of \( X \). Hence \((F, A)\) is a soft BCI-algebra over \( X \). Let \((G, I)\) be a soft set over \( X \) where \( I = \{a,b\} \subset A \) and \( G : I \to \Psi(X) \) be a set-valued function defined by \( G(x) = \{y \in X | y \ast x \preceq 0\} \) for all \( x \in I \). Then \( G(a) = [0,a] \preceq X = F(a), G(b) = [0,a,b] \preceq [0,a,c,d] = F(b) \). Hence \((G, I)\) is a soft ideal of \((F, A)\) but it is not a soft BCI-commutative ideal of \((F, A)\) because \( G(a) \) is not a BCI-commutative ideal of \( X \) related to \( F(a) \) since \((b \preceq c) \preceq a = 0 \in G(a) \) and \( a \in G(a) \) but \( b \preceq (c \preceq c \preceq b) \preceq 0 \in G(a) \).

**Theorem 4.10:** Let \((F, A)\) be a soft BCI-algebra over \( X \). For any soft sets \((G, I)\) and \((H, J)\) over \( X \) we have \((G,I) \ast_{bci-c} (F,A), (H,J) \ast_{bci-c} (F,A) \Rightarrow (G,I) \ast_{bci-c} (H,J) \ast_{bci-c} (F,A)\).

**Proof:** Straightforward. \( \square \)

**Theorem 4.11:** Let \((F, A)\) be a soft BCI-algebra over \( X \). For any soft sets \((G, I)\) and \((H, J)\) over \( X \) in which \( I \) and \( J \) are disjoint, we have \((G,I) \ast_{bci-c} (F,A), (H,J) \ast_{bci-c} (F,A) \Rightarrow (G,I) \ast_{bci-c} (H,J) \ast_{bci-c} (F,A)\).

**Proof:** Assume that \((G,I) \ast_{bci-c} (F,A)\) and \((H,J) \ast_{bci-c} (F,A)\). By means of Definition 3.3, we can write \((G,I) \ast_{bci-c} (H,J) = (R,I) \ast_0 (R,J)\), where \( U = I \cup J \) and for every \( e \in U\),

\[
R(e) = \begin{cases} 
G(e) & \text{if } e \in I \setminus J \\
H(e) & \text{if } e \in J \setminus I \\
G(e) \cup H(e) & \text{if } e \in I \cap J 
\end{cases}
\]

Since \( I \cap J = \emptyset \), either \( e \in I \setminus J \) or \( e \in J \setminus I \) for all \( e \in U \). If \( e \in I \setminus J \), then \( R(e) = G(e) \ast_{bci-c} F(e) \) since \((G,I) \ast_{bci-c} (F,A)\). If \( e \in J \setminus I \), then \( R(e) = H(e) \ast_{bci-c} F(e) \), since \((H,J) \ast_{bci-c} (F,A)\). Thus \( R(e) \ast_{bci-c} F(e) \) for all \( e \in U \) and so, \((G,I) \ast_{bci-c} (H,J) = (R,I) \ast_0 (R,J) \ast_{bci-c} (F,A)\). \( \square \)

If \( I \) and \( J \) are not disjoint in Theorem 4.11, then Theorem 4.11 is not true in general as seen in the following example.

**Example 4.12:** Let \( X = \{0,a,b,c,d\} \) be a BCK-algebra and hence a BCI-algebra, with the following Cayley table:

<table>
<thead>
<tr>
<th>( \ast )</th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>d</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>b</td>
<td>0</td>
<td>0</td>
<td>d</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>c</td>
<td>c</td>
<td>0</td>
<td>d</td>
</tr>
<tr>
<td>d</td>
<td>d</td>
<td>d</td>
<td>d</td>
<td>d</td>
<td>d</td>
</tr>
</tbody>
</table>

Let \((F, A)\) be a soft set over \( X \), where \( A = X \) and \( F : A \to \Psi(X) \) is a set-valued function defined by \( F(x) = \{y \in X | y \ast x \preceq 0\} \), for all \( x \in A \). Then \( F(0) = X, F(a) = F(b) = \emptyset \) and \( F(c) = F(d) = \emptyset \), which are subalgebras of \( X \). Hence \((F, A)\) is a soft BCI-algebra over \( X \). Let \((G, I)\) be a soft set over \( X \) where \( I = \{0,b,c,d\} \subset A \) and \( G : I \to \Psi(X) \) be a set-valued function defined by \( G(x) = \{y \in X | y \ast x = 0\} \) for all \( x \in I \). Then \( G(0) = X, G(a) = \emptyset \) and \( G(b) = \emptyset \) are subalgebras of \( X \). Hence \((G, I)\) is a soft set over \( X \). Since \((G,I) \ast_{bci-c} (F,A)\) and \( (G,b) \ast_{bci-c} (F,A) \), it follows that \( G(e) = G(e) \ast_{bci-c} F(e) \) or \( G(e) = G(e) \ast_{bci-c} F(e) \) for all \( e \in I \). Hence \((G,I) \ast_{bci-c} (F,A)\). This completes the proof. \( \square \)
Hence \((G, I)\) is a soft BCI-commutative ideal of \((F, A)\). Now consider \(J = \{ b \}\) which is not disjoint with \(I\) and let \(H: J \to \Psi(X)\) be a set-valued function by, \(H(y) = \{ x \in X \mid y^*(x^* y) = 0 \}\), for all \(x \in J\). Then \(H(b) = \{0, c\} \sqcup \{0, b, c, d\} = F(b)\). Hence \((H, J)\) is a soft BCI-commutative ideal of \((F, A)\). But if \((R, I) = (G, X) \cup (H, J)\), then \(R(b) = G(b) \cup H(b) = \{0, a, b, c\}\), which is not a BCI-commutative related to \(F(b)\) since \(\{0\} = \{0\}\) is not a BCI-commutative idealistic soft BCI-algebra over \(F(b)\) related to \((F, A)\).

**BCI-Commutative idealistic soft BCI-Algebras**

**Definition 5.1 (Jun and Park [20]):** Let \((F, A)\) be soft set over \(X\). Then \((F, A)\) is called an idealistic soft BCI-algebra over \(X\) if \((F, A)\) is an ideal of \(X\) for all \(x \in A\).

**Definition 5.2:** Let \((F, A)\) be soft set over \(X\). Then \((F, A)\) is called a BCI-commutative idealistic soft BCI-algebra over \(X\) if \(F(x)\) is a BCI-commutative ideal of \(X\) for all \(x \in A\).

**Example 5.3:** Consider a BCI-algebra \(X = \{0, a, b, c, d\}\), which is given in Example 4.8. Let \((F, A)\) be a soft set over \(X\), where \(A = X\) and \(F: A \to \Psi(X)\) is a set-valued function defined by:

\[
F(x) = \begin{cases} 
Z(0,0) & \text{if } x \in \{b, c, d\} \\
X & \text{if } x \in \{0, a\}
\end{cases}
\]

where \(Z(0,0) = \{ x \in X \mid 0^*(0^* x) \in \{0, a\} \}\). Then \((F, A)\) is a BCI-commutative idealistic soft BCI-algebra over \(X\).

For any element \(X\) of a BCI-algebra \(X\), we define the order of \(x\), denoted by \(o(x)\), as \(o(x) = \min\{n \in N \mid 0^* x^* = 0\}\), where \(0^* x^* = \ldots (0^* x^* x_2^* x_1^*)^* x\), in which \(x\) appears \(n\)-times.

**Example 5.4:** Let \(X = \{0, a, b, c, d, e, f, g\}\) be a BCI-algebra defined by the following Cayley table:

\[
\begin{array}{cccccccc}
 & 0 & a & b & c & d & e & f & g \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
a & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
b & b & b & 0 & 0 & 0 & 0 & 0 & 0 \\
c & c & b & a & 0 & 0 & 0 & 0 & 0 \\
d & d & d & d & d & 0 & 0 & 0 & 0 \\
e & e & e & d & d & a & 0 & 0 & 0 \\
f & f & f & f & d & b & b & 0 & 0 \\
g & g & g & g & e & c & b & a & 0
\end{array}
\]

Let \((F, A)\) be a soft set over \(X\), where \(A = \{a, b, c\} \subseteq X\) and \(F: A \to \Psi(X)\) is a set-valued function defined by, \(F(x) = \{ y \in X \mid o(x) = o(y) \}\), for all \(x \in A\). Then \(F(a) = F(b) = F(c) = \{0, a, b, c\}\) is a BCI-commutative ideal of \(X\).

**Example 5.5:** Consider a BCI-algebra \(X = \{0, a, b, c\}\) which is given in Example 4.3. Let \((F, A)\) be a soft set over \(X\), where \(A = X\) and \(F: A \to \Psi(X)\) is a set-valued function defined by, \(F(x) = \{ y \in X \mid y^*(x^* y) = 0 \}\), for all \(x \in A\). Then \(F(0) = \{0\}, F(a) = \{a\}, F(b) = \{0, b\}, F(c) = \{0, c\}\), which are BCI-commutative ideals of \(X\). Hence \((F, A)\) is a BCI-commutative idealistic soft BCI-algebra over \(X\).

Obviously, every BCI-commutative idealistic soft BCI-algebra over \(X\) is an idealistic soft BCI-algebra over \(X\), but the converse is not true in general as seen in the following example.

**Example 5.6:** Consider a BCI-algebra \(X = Y \times Z\), where \((Y, 0^*)\) is a BCI-algebra and \((Z, 0)\) is the adjoint BCI-algebra of the additive group \((Z, +, 0)\) of integers. Let \(F: X \to \Psi(X)\) be a set-valued function defined as follows:

\[
F(y, n) = \begin{cases} 
Y \times N \quad & \text{if } n \in N \\
\{0,0\} & \text{otherwise}
\end{cases}
\]

for all \((y, n) \in X\), where \(N\) is the set of all non-negative integers. Then \((F, X)\) is an idealistic soft BCI-algebra over \(X\) but it is not a BCI-commutative idealistic soft BCI-algebra over \(X\), because \(\{0,0\}\) may not be a BCI-commutative ideal of \(X\).

**Proposition 5.7:** Let \((F, A)\) and \((F, B)\) be soft sets over \(X\) where \(B \subseteq A \subseteq X\). If \((F, A)\) is a BCI-commutative idealistic soft BCI-algebra over \(X\), then so is \((F, B)\).

**Proof:** Straightforward. □

The converse of Proposition 5.7 is not true in general as seen in the following example.

**Example 5.8:** Consider a BCI-commutative idealistic soft BCI-algebra over \(X\) which is described in Example 5.4. If we take \(B = \{a, b, c, d\} \supseteq A\), then \((F, B)\) is not a BCI-commutative idealistic soft BCI-algebra over \(X\) since \(F(d) = \{d, e, f, g\}\) is not a BCI-commutative ideal of \(X\).
Theorem 5.9: Let \((F, A)\) and \((G, B)\) be two BCI-commutative idealistic soft BCI-algebras over \(X\). If \(A \cap B \neq \emptyset\), then the intersection \((F, A) \cap (G, B)\) is a BCI-commutative idealistic soft BCI-algebra over \(X\).

Proof: Using Definition 3.2, we can write, \((F, A) \cap (G, B) = (H, C)\), where \(C = A \cap B\) and \(H(e) = F(e)\) or \(G(e)\) for all \(e \in C\). Note that \(H: C \rightarrow \mathcal{P}(X)\) is a mapping, therefore \((H, C)\) is a soft set over \(X\). Since \((F, A)\) and \((G, B)\) are BCI-commutative idealistic soft BCI-algebras over \(X\), it follows that \(H(e) = F(e)\) is a BCI-commutative ideal of \(X\) or \(H(e) = G(e)\) is a BCI-commutative ideal of \(X\) for all \(e \in C\). Hence \((H, C) = (F, A) \cap (G, B)\) is a BCI-commutative idealistic soft BCI-algebra over \(X\). □

Corollary 5.10: Let \((F, A)\) and \((G, B)\) be two BCI-commutative idealistic soft BCI-algebras over \(X\). Then their intersection \((F, A) \cap (G, B)\) is a BCI-commutative idealistic soft BCI-algebra over \(X\).

Proof: Straightforward. □

Theorem 5.11: Let \((F, A)\) and \((G, B)\) be two BCI-commutative idealistic soft BCI-algebras over \(X\). If \(A\) and \(B\) are disjoint, then the union \((F, A) \cup (G, B)\) is a BCI-commutative idealistic soft BCI-algebra over \(X\).

Proof: By means of Definition 3.3, we can write, \((F, A) \cup (G, B) = (H, C)\), where \(C = A \cup B\) and for every \(e \in C\),

\[
H(x) = \begin{cases} 
F(e) & \text{if } e \in A \setminus B \\
G(e) & \text{if } e \in A \setminus B \\
F(e) \cup G(e) & \text{if } e \in A \cap B 
\end{cases}
\]

Since \(A \cap B \neq \emptyset\), either \(e \in A \setminus B\) or \(e \in B \setminus A\) for all \(e \in C\). If \(e \in A \setminus B\), then \(H(e) = F(e)\) is a BCI-commutative ideal of \(X\), since \((F, A)\) is a BCI-commutative idealistic soft BCI-algebra over \(X\). If \(e \in B \setminus A\), then \(H(e) = G(e)\) is a BCI-commutative ideal of \(X\) since \((G, B)\) is a BCI-commutative idealistic soft BCI-algebra over \(X\). Hence \((H, C) = (F, A) \cup (G, B)\) is a BCI-commutative idealistic soft BCI-algebra over \(X\). □

Theorem 5.12: Let \((F, A)\) and \((G, B)\) be two BCI-commutative idealistic soft BCI-algebras over \(X\), then \((F, A) \cap (G, B)\) is a BCI-commutative idealistic soft BCI-algebra over \(X\).

Proof: By means of Definition 3.4, we know that, \((F, A) \cap (G, B) = (H, C)\), where \(H(x, y) = F(x) \cap G(y)\) for all \((x, y)\) \(\in A \times B\). Since \(F(x)\) and \(G(y)\) are BCI-commutative ideals of \(X\), the intersection \(F(x) \cap G(y)\) is also a BCI-commutative ideal of \(X\). Hence \(H(x, y)\) is a BCI-commutative ideal of \(x\) for all \((x, y) \in A \times B\). Hence \((F, A) \cap (G, B) = (H, C)\) is a BCI-commutative idealistic soft BCI-algebra over \(X\). □

Definition 5.13: A BCI-commutative idealistic soft BCI-algebra \((F, A)\) over \(X\) is said to be trivial (resp., whole) if \(F(x) = 0\) (resp., \(F(x) = X\)) for all \(x \in A\).

Example 5.14: Let \(X\) be a BCI-algebra which is given in Example 4.3 and let \(f: X \rightarrow \mathcal{P}(Y)\) be a set-valued function defined by, \(f(x) = \emptyset \cup \{y \in Y \mid \phi(x) = \phi(y)\}\), for all \(x \in X\). Then \(f(0) = \{0\}\) and \(f(a) = f(b) = f(c) = X\), which are BCI-commutative ideals of \(X\). Hence \((F, \{0\})\) is a trivial BCI-commutative idealistic soft BCI-algebra over \(X\) and \((F, X \setminus \{0\})\) is a whole BCI-commutative idealistic soft BCI-algebra over \(X\).

The proofs of the following three lemmas are straightforward, so they are omitted.

Lemma 5.15: Let \(f: X \rightarrow Y\) be an onto homomorphism of BCI-algebras. If \(I\) is an ideal of \(X\), then \(f(I)\) is an ideal of \(Y\).

Lemma 5.16: Let \(f: X \rightarrow Y\) be an isomorphism of BCI-algebras. If \(I\) is a BCI-commutative ideal of \(X\), then \(f(I)\) is a BCI-commutative ideal of \(Y\).

Lemma 5.17: Let \(f: X \rightarrow Y\) be an isomorphism of BCI-algebras. If \((F, A)\) is a BCI-commutative idealistic soft BCI-algebra over \(X\), then \((f(F), A)\) is a BCI-commutative idealistic soft BCI-algebra over \(Y\).

Theorem 5.18: Let \(f: X \rightarrow Y\) be an isomorphism of BCI-algebras and let \((F, A)\) be a BCI-commutative idealistic soft BCI-algebra over \(X\).

- If \(F(x) = \ker(f)\) for all \(x \in A\), then \((f(F), A)\) is a trivial BCI-commutative idealistic soft BCI-algebra over \(Y\).
- If \((F, A)\) is whole, then \((f(F), A)\) is a whole BCI-commutative idealistic soft BCI-algebra over \(Y\).

Proof: (1) Assume that \(F(x) = \ker(f)\) for all \(x \in A\). Then \(f(F(x)) = f(\{0\})\) for all \(x \in A\). Hence \((F, A)\) is a trivial BCI-commutative idealistic soft BCI-algebra over \(Y\) by Lemma 5.17 and Definition 5.13.
Suppose that \((F, A)\) is whole. Then \(F(x) = X\) for all \(x \in A\) and so \(f(F)(x) = f(F(x)) = f(X) = Y\) for all \(x \in A\). It follows from Lemma 5.17 and Definition 5.13 that \((f(F), A)\) is a whole BCI-commutative idealistic soft BCI-algebra over \(Y\).\(\Box\)

**Definition 5.19 (Jun and Meng [21]):** A fuzzy set \(\mu\) in \(X\) is called a fuzzy BCI-commutative ideal of \(X\), if for all \(x, y, z \in X\),

(i) \(\mu(0) \geq \mu(x)\)

(ii) \(\mu((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))) \geq \min\{\mu((x \ast y) \ast z), \mu(z)\}\)

The transfer principle for fuzzy sets described in [22] suggest the following theorem.

**Lemma 5.20 (Jun and Meng [21]):** A fuzzy set \(\mu\) in \(X\) is a fuzzy BCI-commutative ideal of \(X\) if and only if for any \(t \in [0,1]\), the level subset \(U\{\mu; t\} = \{x \in X | \mu(x) \geq t\}\) is either empty or a BCI-commutative ideal of \(X\).

**Theorem 5.21:** For every fuzzy BCI-commutative ideal \(\mu\) of \(X\), there exists a BCI-commutative idealistic soft BCI-algebra \((F, A)\) over \(X\).

**Proof:** Let \(\mu\) be a fuzzy BCI-commutative ideal of \(X\). Then \(U\{\mu; t\} = \{x \in X | \mu(x) \geq t\}\) is an BCI-commutative ideal of \(X\) for all \(t \in Im(\mu)\). If we take \(A = Im(\mu)\) and consider a set valued function \(F: A \rightarrow \Psi(X)\) given by \(F(t) = U\{\mu; t\}\), for all \(t \in A\), then \((F, A)\) is a BCI-commutative idealistic soft BCI-algebra over \(X\).\(\Box\)

Conversely, the following theorem is straightforward.

**Theorem 5.22:** For any fuzzy set \(\mu\) in \(X\), if a BCI-commutative idealistic soft BCI-algebra \((F, A)\) over \(X\) is given by \(A = Im(\mu)\) and \(F(t) = U\{\mu; t\}\) for all \(t \in A\), then \(\mu\) is a fuzzy BCI-commutative ideal of \(X\).

Let \(\mu\) be a fuzzy set in \(X\) and let \((F, A)\) be a soft set over \(X\) in which \(A = Im(\mu)\) and \(F: A \rightarrow \Psi(X)\) is a set-valued function defined by

\[
F(t) = \{x \in X | \mu(x) + t > 1\}
\]

for all \(t \in A\). Then there exists \(t \in A\) such that \(F(t)\) is not a BCI-commutative ideal of \(X\) as seen in the following example.

**Example 5.23:** For any BCI-algebra \(X\), define a fuzzy set \(\mu\) in \(X\) by \(\mu(0) = 0.5\) and \(\mu(x) = 1 - x\) for all \(x \neq 0\). Let \(A = Im(\mu)\) and \(F: A \rightarrow \Psi(X)\) be a set-valued function defined by(5.2). Then \(F(1 - x) = X \setminus \{0\}\), which is not a BCI-commutative ideal of \(X\).\(\Box\)

**Theorem 5.24:** Let \(\mu\) be a fuzzy set in \(X\) and let \((F, A)\) be a soft set over \(X\) in which \(A = [0,1]\) and \(F: A \rightarrow \Psi(X)\) is given by (5.2). Then the following assertions are equivalent:

- \(\mu\) is a fuzzy BCI-commutative ideal of \(X\).
- For every \(t \in A\) with \(F(t) \neq \emptyset\), \(F(t)\) is an BCI-commutative ideal of \(X\).

**Proof:** Assume that \(\mu\) is a fuzzy BCI-commutative ideal of \(X\). Let \(t \in A\) be such that \(F(t) \neq \emptyset\). Then for any \(x \in F(t)\), we have \(\mu(0) + t \geq \mu(x) + t > 1\), that is, \(0 \in F(t)\). Let \((x \ast y) \ast z \in F(t)\) for any \(x, y, z \in X\). Then \(\mu((x \ast y) \ast z) + t > 1\) and \(\mu(z) + t > 1\). Since \(\mu\) is a fuzzy BCI-commutative ideal of \(X\), it follows that,

\[
\mu((x \ast (y \ast (y \ast x))) \ast (0 \ast (0 \ast (x \ast y)))) + t \geq \min\{\mu((x \ast y) \ast z), \mu(z)\} + t, \mu(z) + t > 1
\]

so that \((x \ast (y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))) \in F(t)\). Hence \(F(t)\) is a BCI-commutative ideal of \(X\) for all \(t \in A\) such that \(F(t) \neq \emptyset\).

Conversely, suppose that (2) is valid. If there exists \(x \in X\) such that \(\mu(0) < \mu(x)\), then there exists \(t \in A\) such that \(\mu(0) + t < \mu(x) + t\). It follows that \(x \in F(t)\) and \(\mu(0) \not\geq \mu(x)\), which is a contradiction. Hence \(0 \not\in F(x)\) for all \(x \in X\). Now assume that,

\[
\mu((x \ast (y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))) < \min\{\mu((x \ast y) \ast z), \mu(z)\}
\]

for some \(x, y, z \in X\). Then there exists some \(s \in A\) such that,

\[
\mu((x \ast (y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))) + s \leq 1 < \min\{\mu((x \ast y) \ast z), \mu(z)\} + s
\]

\[
\Rightarrow \mu((x \ast (y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))) + s \leq 1 < \min\{\mu((x \ast y) \ast z), \mu(z) + s\}
\]

which implies that \((x \ast y) \ast z \in F(s)\) and \(s \in F(x)\) but \(x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))) \not\in F(s)\).

This is a contradiction. Therefore, \(\mu((x \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))) \geq \min\{\mu((x \ast y) \ast z), \mu(z)\}\), for all \(x, y, z \in X\) and thus \(\mu\) is fuzzy BCI-commutative ideal of \(X\).\(\Box\)
**Corollary 5.25**: Let \( \mu \) be a fuzzy set in \( X \) such that \( \mu(x) > 0.5 \) for all \( x \in X \) and let \((F, A)\) be a soft set over \( X \) in which, \( A := \{ t \in \text{Im}(\mu) \mid t > 0.5 \} \) and \( \mathcal{F}: A \rightarrow \mathcal{P}(X) \) is given by (5.2). If \( \mu \) is a fuzzy BCI-commutative ideal of \( X \), then \((F, A)\) is a BCI-commutative idealistic soft BCI-algebra over \( X \).

**Proof**: Straightforward. \( \square \)

**Theorem 5.26**: Let \( \mu \) be a fuzzy set in \( X \) and let \((F, A)\) be a soft set over \( X \) in which \( A = (0.5, 1) \) and \( \mathcal{F}: A \rightarrow \mathcal{P}(X) \) is defined by, \( \mathcal{F}(t) = \mu(t) \) for all \( t \in A \). Then \( F(t) \) is a BCI-commutative ideal of \( X \) for all \( t \in A \) with \( F(t) \neq \emptyset \) if and only if the following assertions are valid:

1. \( \max(\mu(0), 0.5) \geq \mu(x) \) for all \( x \in X \)
2. \( \max(\mu((x^*y^*)(y^*)^*z^*0^*(0^*(x^*y)))), 0.5) \geq \min(\mu((x^*y^*)), 0.5) \) for all \( x, y, z \in X \).

**Proof**: Assume that \( F(t) \) is a BCI-commutative ideal of \( X \) for all \( t \in A \) with \( \mathcal{F}(t) \neq \emptyset \). If there exists \( x \in X \) such that \( \max(\mu(0), 0.5) \leq \mu(x) \), then there exists \( t \in A \) such that \( \max(\mu(0), 0.5) \leq \mu(x) \). It follows that \( \mu(x) \geq 0.5 \), so that \( \mu(x) > 0.5 \). This is a contradiction. Therefore (1) is valid. Suppose that there exist \( a, b, c \in X \) such that \( \max(\mu((b^*a^*)^*0^*(0^*(a^*b)))), 0.5) \geq \min(\mu((a^*b^*)^*c^*)), 0.5) \). Then there exists \( x \in X \) such that

\[
\max(\mu(a^*b^*c), 0.5) \geq \min(\mu(a^*b^*c), 0.5)
\]

which implies \( (a^*b^*)^*c \in \mathcal{F}(w) \) and \( c \in \mathcal{F}(w) \) but \( a^*b^* \notin \mathcal{F}(w) \). This is a contradiction. Hence (2) is valid.

Conversely, suppose that (1) and (2) are valid. Let \( t \in A \) with \( \mathcal{F}(t) \neq \emptyset \). Then for any \( x \in F(t) \), we have:

\[
\max(\mu(0), 0.5) \geq \mu(x) \geq t > 0.5
\]

which implies \( \mu(0) \geq t \) and thus \( 0 \in F(t) \). Let \( x^*y^*z \in F(t) \) and \( z \in F(t) \), for any \( x, y, z \in X \). Then \( \mu((x^*y^*z)) \geq t \) and \( \mu(z) \geq t \). It follows from the second condition that

\[
\max(\mu((x^*y^*z)^*0^*(0^*(x^*y))), 0.5) \geq \min(\mu((x^*y^*z)), 0.5) \geq t > 0.5
\]

so that \( x^*((y^*(y^*x))^*0^*(0^*(x^*y))) \in \mathcal{F}(t) \). Therefore \( F(t) \) is a BCI-commutative ideal of \( X \) for all \( t \in A \) with \( F(t) \neq \emptyset \).

\( \square \)

**CONCLUSION**

The concept of soft set, which is introduced by Molodtsov [19], is a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. Soft sets are deeply related to fuzzy sets and rough sets. We introduced the notion of soft BCI-commutative ideals and BCI-commutative idealistic soft BCI-algebras and discussed related properties. We established the intersection, union, “AND” operation and “OR” operation of soft BCI-commutative ideals and BCI-commutative idealistic soft BCI-algebras. From the above discussion it can be observed that fuzzy BCI-commutative ideals can be characterized using the concept of soft sets. For a soft set \((F, A)\) over \( X \), a fuzzy set \( \mu \) in \( X \) is a fuzzy BCI-commutative ideal of \( X \) if and only if for every \( t \in A \) with \( \mathcal{F}(t) \neq \emptyset \), \( F(t) \) is a BCI-commutative ideal of \( X \). Finally we have discussed the relations between fuzzy BCI-commutative ideals and BCI-commutative idealistic soft BCI-algebras.

**REFERENCES**


