Boundary Value Problems in the Theory of Oscillations of Rectangular Plates Interacting with a Deformable Medium

A. Zh. Seitmuratov, U. Umbetov and U. Zh. Aitimova

The Korkyt Ata Kyzylorda State University, Kazakhstan
Taraz State University named after M. Kh. Dulaty, Taraz, Kazakhstan

Abstract: In this work develops a theory of vibrations of laminated plates of building structures, strictly justified by the staging of various boundary value problems of oscillation. In the study of oscillations of plates accurate three-dimensional problem is replaced by a simpler, two-dimensional points of median plane of the plate, which imposes limitations on the external conditions. These restrictions are that external forces cannot be high frequency. As a general equation of oscillations of plates, obtained before contain derivatives of any order of the coordinates \(x, y\) and time \(t\), stacked on the structure and therefore are not suitable for solving applied tasks and of engineering calculations. For this purpose it is necessary to formulate approximate boundary value problems of oscillation.

Key words: Oscillation • Plate • Deformable medium • Elastic and viscoelastic medium

INTRODUCTION

The materials used in building structures have elastic and viscoelastic properties, are anisotropic, multilayer and other mechanical characteristics.

Two-dimensional elements are components of many structures. Construction of general and approximate equations of oscillations of a various kind of plane elements is an actual problem in the development of theoretical bases of calculation of building designs and construction in general. These problems include the objectives of improved models of non-stationary nature of the structures and their elements, the materials of which are difficult mechanical, rheological properties inherent to various building designs under the influence of various external factors [1-12].

The result of the study [13] shows that the mechanical and geometrical characteristics are the main elements of the seismic stability of many structures.

In this paper develops a theory of vibrations of laminated plates of building structures, strictly justified by the staging of various boundary value problems of oscillation.

The Main Part: Let limitless in terms of plate thickness \(2h\) is below the surface of a semi-infinite medium at depth \((h_0-h_1)\). Put the plane \(XY\) in the median plane of the plate at \(z=0\). Axis \(OZ\) to the right towards the outer surface of the outer layer. Denote the layer parameters index”1”, the upper layer \([-\infty<(x,y)<\infty; h_1 \leq z \leq (h_0-h_1)]\) by the index”2” and the lower half-space \([-\infty<(x,y)<\infty; -h \leq z \leq 0]\) - index”3”.

We assume that the materials of the upper layer of the plate and the grounds are homogeneous, isotropic, show the viscous properties.

We introduce the potentials of \(\Phi(l)\) and \(\Psi(l)\) transverse March 25, 2014and longitudinal waves well-known formulas

\[
\dot{u}(l) = \text{grad}\Phi(l) + \text{rot}\Psi(l)
\]

where \(\dot{u}(l)\) – the displacement vector points in the layer of the plate and the foundation.

In potentials \(\Phi(l)\) and \(\Psi(l)\) equations of motion of a layer of the plate and the foundation will take the form of:

\[
N(l)\left(\Delta\Phi(l)\right) = \rho_l \frac{\partial^2\Phi(l)}{\partial t^2}
\]
\[
M(l)\left(\Delta\Psi(l)\right) = \rho_l \frac{\partial^2\Psi(l)}{\partial t^2}
\]

where the operator \(N(l)\) is equal to:

Corresponding Author: A. Zh. Seitmuratov, The Korkyt Ata Kyzylorda State University, Kazakhstan.

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\[ N_l = L_l + 2M_l \]

\[ \Delta - \text{three-dimensional Laplace operator} \]

\[ \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \]

\[ i_l, M_l - \text{viscoelastic operators.} \]

The Helmholtz theorem, in the absence of internal sources of the vector potential \( \varphi \) of transverse waves must satisfy the condition:

\[ \text{div} \varphi(l) = 0 \quad (3) \]

a closing equation for finding four unknown potentials \( \varphi(l), \psi_1(l), \psi_2(l), \psi_3(l) \).

Displacement, \( u,v,w \) deformation \( \varepsilon \), and strains in Cartesian coordinates through the potentials \( \Phi \) and \( \Psi \) the longitudinal and transverse waves are determined by the well-known formulas.

In (1) it is shown that the boundary value problem vibrations of the plate under the surface are reduced to the integro-differential equations (2) with boundary and initial conditions: on the outer surface \( z = h \)

\[ \sigma_{zz}^{(2)} = f_z^{(2)}(x,y,t); \sigma_{zz}^{(2)} = f_{zz}^{(2)}(x,y,t); \quad (4) \]

on the contact boundary between the upper layer – plate \( z = h \)

\[ \sigma_{zz}^{(2)} = f_z^{(2)}(x,y,t); \sigma_{zz}^{(2)} = f_{zz}^{(2)}(x,y,t); \quad (5) \]

on the boundary of the plate - foundation \( z = -h \)

\[ \sigma_{zz}^{(1)} = \sigma_{zz}^{(3)} + f_{zz}^{(3)}(x,y,t); \]

\[ \sigma_{zz}^{(1)} = 0; \sigma_{zz}^{(3)} + f_{zz}^{(3)}(x,y,t) = 0; \quad (6) \]

\[ u(l) = w(l) + f_0^{(3)}(x,y,t)(j = x,y) \]

Furthermore, there should be conditions decay at infinity, i.e. at \( z \rightarrow \infty \)

\[ \varphi^{(3)} = 0; \quad \psi_1^{(3)} = \psi_2^{(3)} = \psi_3^{(3)} = 0 \quad (7) \]

The initial conditions are zero, i.e.

\[ \varphi(l) = \frac{\partial \varphi(l)}{\partial t} = \frac{\partial \psi(l)}{\partial t} = 0 \quad (l = 1,3); t = 0 \quad (j = 1,2,3) \quad (8) \]

The task of the vibrations of the plate in a differentiable environment is reduced to the study of equation (2) satisfying the boundary (4), (5), (6) and initial conditions (8).

In the study of oscillations of plate’s accurate three-dimensional problem is replaced by simpler, two-dimensional points of median plane of the plate, which imposes limitations on the external conditions. These restrictions are that external forces cannot be high frequency.

The above problem is solved with the use of Fourier \( X \) and \( Y \) Laplace at \( t \).

In the work [1] found the General solution of the formulated three-dimensional tasks for zero initial conditions and the General expressions for the displacement and strain.
where operators \( \lambda_1^{(1)} \) and \( \lambda_2^{(1)} \) are equal

\[
\lambda_1^{(1)} = \left[ \frac{\partial M_1}{\partial x} \right] - \Delta, \quad \lambda_2^{(1)} = \left[ \frac{\partial M_1}{\partial y} \right] - \Delta.
\]

\[
\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},
\]

\[
\sigma_{xx}^{(1)} = M_1 \sum_{n=0}^{\infty} \left[ (1-C_1) \lambda_2^{(n)} + C_1 Q_{in} \left( \lambda_2^{(n)} - 2 \lambda_2^{(1)} \right) + \frac{\partial^2}{\partial x^2} \left( \frac{\partial U^{(1)}}{\partial x} \right) + \frac{\partial^2}{\partial y^2} \left( \frac{\partial V^{(1)}}{\partial x} \right) - \frac{\partial^2}{\partial x^2} \left( \frac{\partial W^{(1)}}{\partial x} \right) \right] \frac{z^{2n}}{(2n+1)!};
\]

\[
\sigma_{yy}^{(1)} = M_1 \sum_{n=0}^{\infty} \left[ (1-C_1) \lambda_2^{(n)} + C_1 Q_{in} \left( \lambda_2^{(n)} - 2 \lambda_2^{(1)} \right) + \frac{\partial^2}{\partial x^2} \left( \frac{\partial U^{(1)}}{\partial y} \right) + \frac{\partial^2}{\partial y^2} \left( \frac{\partial V^{(1)}}{\partial y} \right) - \frac{\partial^2}{\partial y^2} \left( \frac{\partial W^{(1)}}{\partial y} \right) \right] \frac{z^{2n}}{(2n+1)!};
\]

\[
\sigma_{zz}^{(1)} = M_1 \sum_{n=0}^{\infty} \left[ (1-C_1) \lambda_2^{(n)} + C_1 Q_{in} \left( \lambda_2^{(n)} - 2 \lambda_2^{(1)} \right) + \frac{\partial^2}{\partial x^2} \left( \frac{\partial U^{(1)}}{\partial z} \right) + \frac{\partial^2}{\partial y^2} \left( \frac{\partial V^{(1)}}{\partial z} \right) - \frac{\partial^2}{\partial y^2} \left( \frac{\partial W^{(1)}}{\partial z} \right) \right] \frac{z^{2n}}{(2n+1)!};
\]
where unknown \( \varphi_l^{(1)}, \varphi_v^{(1)} \) are the tangent and normal displacements of the points in the plane \( z=0 \) and points of median plane of the plate, \( \varphi_l^{(1)}, \varphi_v^{(1)}, w^{(1)} \) - value of derivatives \( Z \) transverse displacement or values type of deformation \( w^{(1)} \) - deformation at \( Z = 0 \).

Operators \( \lambda_1, \lambda_2 \) are two-dimensional integro-differential describing the propagation of longitudinal and transverse waves in a plane \( z=0 \).

For finding the unknown \( \varphi_l^{(1)}, \varphi_v^{(1)}, w^{(1)} \) we have boundary conditions (4)-(6)

Using the formulas (9) and (11) of strains and displacements substituting these expressions into the boundary conditions (4)-(6) equations are obtained for determining the unknown functions that are general solutions of the considered problem and describing oscillations of a three-dimensional environment.

For the study of oscillations of rectangular plates in terms of the need to formulate boundary problems.

Under the boundary tasks fluctuations limited plate in the plan, which is under the surface, means the conclusion of the equation of oscillations of plates; the formulation of boundary conditions on the edges of the plate and the initial conditions for the functions.

As a general equations of oscillations of plates obtained by the author [1] contain derivatives of any order of the coordinates \( x, y, t \), stacked on the structure and therefore are not suitable for solving applied tasks and of engineering calculations.

For this purpose it is necessary to formulate approximate boundary value problems of oscillation.

In [2-5] is obtained approximate equation of transverse vibrations of the plate for transverse shift \( w^{(1)} \) points of median plane of the plate in the shape of

\[

\begin{align*}
\varphi_l^{(1)} &= -2D_{ll}Q_{ln} \frac{\partial^2 \varphi_l^{(1)}}{\partial y^2} - \left[ D_{ll}Q_{nn} (\lambda_2^{(2)} - \Delta) - \\
- \lambda_1^{(n)} \frac{\partial^2 \varphi_l^{(1)}}{\partial y^2} \right] \frac{z^{2n}}{(2n)!} \\
\varphi_v^{(1)} &= -2D_{vv}Q_{nn} \frac{\partial^2 \varphi_v^{(1)}}{\partial y^2} - \left[ D_{vv}Q_{nn} (\lambda_2^{(2)} - \Delta) - \\
- \lambda_1^{(n)} \frac{\partial^2 \varphi_v^{(1)}}{\partial y^2} \right] \frac{z^{2n}}{(2n)!} \\
w^{(1)} &= -2D_{ww}Q_{nn} \frac{\partial^2 w^{(1)}}{\partial y^2} - \left[ D_{ww}Q_{nn} (\lambda_2^{(2)} - \Delta) - \\
- \lambda_1^{(n)} \frac{\partial^2 w^{(1)}}{\partial y^2} \right] \frac{z^{2n}}{(2n)!} \\
\end{align*}

\]

\( \lambda_1^{(n)} \) are the tangent and normal displacements of the points in the plane \( z=0 \) and points of median plane of the plate.

For finding the unknown \( \varphi_l^{(1)}, \varphi_v^{(1)}, w^{(1)} \) we have boundary conditions (4)-(6)

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\begin{align*}
A_1 &= \rho_1 M^{-1} \varphi_l^{(1)} + \rho_2 N_1^{(1)} (h_0 - h_1); \\
A_2 &= \rho_1^2 M^{-1} \left( N_1^{(1)} + 3M_1^{(1)} \right) \frac{\partial^2 \varphi_l^{(1)}}{\partial t^2} + \rho_2 N_1^{(1)} \varphi_l^{(1)} \times \\
&\times \left( \frac{z^{2n}}{(2n)!} \right) \\
A_3 &= \left( 3 - 4M_2^{(1)} - 2N_1^{(1)} \right) \left( \rho_1 \left( M_1^{(1)} + 3N_1^{(1)} \right) \frac{\partial^2 \varphi_l^{(1)}}{\partial t^2} \right) + \\
&\times \rho_2 N_1^{(1)} \frac{z^{2n}}{(2n)!} \left( \varphi_l^{(1)} \right) \\
\Phi(x, y, t) &= \left| 1 - \left( 3 - 2M_1^{(1)} \right) \right| \left( \frac{\partial^2 \varphi_l^{(1)}}{\partial t^2} \right) + \\
&\times \left( \frac{z^{2n}}{(2n)!} \right) \\
&= \left( M_1^{(1)} \right) \left( \rho_1 \left( M_1^{(1)} + 3N_1^{(1)} \right) \frac{\partial^2 \varphi_l^{(1)}}{\partial t^2} \right) + \\
&\times \left( \frac{z^{2n}}{(2n)!} \right) \\
\end{align*}

\]

Reaction of the basis of \( P \), is determined by the formula (13) includes both the velocity of the points transverse displacement of the plane \( z=0 \) and the odd derivatives time.

Thus, the law of resistance \( \rho M \) (13) explicitly contains the parameters of the plate, foundation and topcoat.

Despite the fact that the equation (12) is approximate, it is quite difficult. In the operators (13) contains all the parameters and operators characterizing the mechanical and rheological properties of the materials of plates - and the base layers and their thicknesses.

We derive the boundary conditions on the edges of a rectangular plate. For simplicity, consider a flat edge \( x = const \), the boundary conditions for easy record of the conditions for \( x = const \) and for an arbitrary curved edge of the known formulas through the boundary conditions at \( x = const \), \( y = const \).

Boundary conditions will display on the theory of thick platform or plates.
Based on the boundary conditions on a surface of a plate \( z = \pm h \) we obtain the dependence of the quantities \( u^{(1)}_i, v^{(1)}_i \), from transverse shift \( w^{(1)}_i \).

\[
u^{(1)}_1 = - \frac{\partial W^{(1)}_1}{\partial x}; \quad v^{(1)}_1 = - \frac{\partial W^{(1)}_1}{\partial y}
\]  

(14)

Rigid fixation of the edge \( x = \text{const} \). As it is known from the theory of thick plates are two types of such fastening

\[
u^{(1)}_1 = v^{(1)}_1 = w^{(1)}_1 = 0
\]  

(15)

\[
u^{(1)}_1 = w^{(1)}_1 = \sigma^{(1)}_{yy} = 0
\]  

(16)

hinge-supported edge \( x = \text{const} \).

For this fix also two ways of fastening.

\[
u^{(1)}_1 = w^{(1)}_1 = \sigma^{(1)}_{xx} = 0
\]  

(17)

\[
u^{(1)}_1 = \sigma^{(1)}_{xx} = \sigma^{(1)}_{yy} = 0
\]  

(18)

Edge, free from strain.

For the free edge stringent conditions are

\[
\sigma^{(1)}_{xx} = \sigma^{(1)}_{yy} = \sigma^{(1)}_{xy} = 0
\]  

(19)

Rigid and hinged fixations are quite simple and using the approximate formulas (13) and the dependence (15), we obtain for the transverse displacement of the boundary conditions:

For rigid fixation

\[
\nu^{(1)}_1 = \frac{\partial W^{(1)}_1}{\partial x} = 0
\]  

(20)

For hinged fixation

\[
\nu^{(1)}_1 = \frac{\partial^2 W^{(1)}_1}{\partial x^2} = 0
\]  

(21)

It is obtained

\[
(2 + 3D_1) \frac{\partial^2 W^{(1)}_1}{\partial x^2} + (1 + D_1) \frac{\partial^2 W^{(1)}_1}{\partial y^2} - \rho M^{-1} \left( \frac{\partial^2 W^{(1)}_1}{\partial t^2} \right) = 0
\]  

(22)

excluding \( \frac{\partial^2 w^{(1)}_1}{\partial t^2} \) in the second condition (22) from the first, we obtain

\[
(2 + 3D_1) \frac{\partial^2 W^{(1)}_1}{\partial x^2} + (1 + D_1) \frac{\partial^2 W^{(1)}_1}{\partial y^2} - \rho (1 + D_1) M^{-1} \left( \frac{\partial^2 W^{(1)}_1}{\partial t^2} \right) = 0
\]  

(23)

The third of the conditions (19) gives

\[
\frac{\partial^3 w^{(1)}_1}{\partial x_1} = F(t),
\]

i.e. in the first approximation \( \frac{\partial w^{(1)}_1}{\partial x} \) does not depend on \( y \) and decide upon a solution to a particular problem.

The first additive component in (23) differs from the classical and the second matches. The first condition (23) takes into account the deformability of the edge over time and d’Alembert principle similar to the dynamics of a material point.

General initial conditions for the plate as a three-dimensional body are of the form:

\[
u^{(1)} = v^{(1)} = w^{(1)} = 0;
\]

\[
\frac{\partial u^{(1)}}{\partial t} = \frac{\partial v^{(1)}}{\partial t} = \frac{\partial w^{(1)}}{\partial t} = 0; (t = 0)
\]  

(24)

Using dependency (14) for movements have

\[
u^{(1)} = - \frac{\partial W^{(1)}_1}{\partial x} z + D_1 \frac{\partial}{\partial x} \Delta W^{(1)}_1 z^3 6;
\]

\[
v^{(1)} = - \frac{\partial W^{(1)}_1}{\partial y} z + D_1 \frac{\partial}{\partial y} \Delta W^{(1)}_1 z^3 6;
\]  

(25)

For the edge, which is free from strain, in the formula of \( \sigma_{xx} \) take the first additive component \( z \) and for \( \sigma^{(1)}_{xx} \) the first two, as the first identical equation is zero due to conditions (15)
In the beginning let us consider the initial conditions from (24) for the movement. Then from the expressions (25) we get

\[ \frac{\partial W_1^{(1)}}{\partial x} = 0; \frac{\partial \Delta W_1^{(1)}}{\partial x} = 0; \]
\[ \frac{\partial W_1^{(1)}}{\partial y} = 0; \frac{\partial \Delta W_1^{(1)}}{\partial y} = 0; \]
\[ W_1^{(1)} = 0; \frac{\partial ^2 W_1^{(1)}}{\partial t^2} = 0 \]

Differentiating the expression (25) and by using the second three initial conditions (24), we similarly obtain

\[ \frac{\partial W_1^{(1)}}{\partial t} = \frac{\partial ^3 W_1^{(1)}}{\partial t^3} = 0 \]

Initial conditions (26) and (27) provide the necessary number of initial conditions for transverse shift \( W_1^{(1)} \) meets with the hyperbolic equation of fourth order in the coordinates and time.

**CONCLUSION**

In conclusion, it should be noted that the evaluation and conclusion initial and boundary conditions for the plate under the surface completely coincides with similar initial boundary conditions for a free plate, obtained in the works [1].

Thus, in the formulation of boundary value problems boundary conditions do not depend on the presence of the upper layer and lower foundation.

**REFERENCES**