Certain New Classes of Meromorphic Functions Associated with Convolution Operator

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Abstract: In this paper, making use of a linear operator we introduce and study certain new classes of meromorphic functions. We derive some inclusion relationships, coefficient bounds and a radius problem. These classes contain many known classes as special cases.

Key words: Meromorphic functions, hadamard product, close-to-convex functions

INTRODUCTION

Let M denotes the class of meromorphic functions of the form

\[ f(z) = \frac{1}{z} + \sum_{j=0}^{\infty} a_j z^j \]  

which are analytic in the punctured open unit disc \( D = \{ z \in \mathbb{C} : 0 < |z| < 1 \} \). Further, let \( P_k(\gamma) \) be the class of functions \( p(z) \) analytic in \( E = D \cup \{0\} \) satisfying

\[ \frac{\int_0^{2\pi} |p(z) - \gamma|}{1 - \gamma} d\theta \leq k \pi \]  

(1.2)

where \( z = re^{\theta}, k \geq 2, 0 \leq \gamma < 1 \). This class was introduced by Padmanabhan and Parvatham [7]. For \( \gamma = 0 \) we obtain the class \( P_k \) defined by Pinchuk [8] and for \( k = 2, P_2(\gamma) = P(\gamma) \) is the class with real part greater than \( \gamma \).

Also from (1.2) it can be seen that \( p \in P_k(\gamma) \) if and only if

\[ p(z) = \frac{k+1}{4} p(z) - \frac{k-1}{2} p(z) \]  

where, \( p, p_2(\gamma) = P(\gamma) \) for \( z \in E \). The class is closed under the convolution (Hadamard product) denoted and defined by

\[ (f * g)(z) = \frac{1}{z} + \sum_{j=0}^{\infty} b_j z^j \]  

In [1] Aouf et al. defined a convolution operator

\[ \theta_k\left( (\alpha_n, A_n), (\beta_n, B_n)_j \right) : M \rightarrow M \] as follows:

\[ \theta_k\left( (\alpha_n, A_n), (\beta_n, B_n)_j \right) f(z) = \frac{1}{z} + \sum_{j=0}^{\infty} \frac{\Gamma(\lambda + j + 1)_{n+1} \Gamma(\beta_n + (j+1)B_n)_j}{\Gamma(\lambda)_n \Gamma(\alpha_n + (j+1)A_n)_j} \]  

(1.3)

where \( \lambda > 0, \in D \). For convenience we write

\[ \theta_k(\alpha_n, A_n) = \theta_k\left( (\alpha_n, A_n), (\beta_n, B_n)_j \right) \]

From (1.3), it can be easily verified that

\[ A \theta_k(\alpha_n, A_n) f(z) = \alpha_n \theta_k(\alpha_n) f(z) - (\alpha_n + A_n) \theta_k(\alpha_n + 1) f(z) \]  

(1.4)

and

\[ z \theta_k(\alpha_n, A_n) f(z) = \lambda \theta_k(\alpha_n) f(z) - \theta_k(\alpha_n, A_n) \theta_k(\alpha_n) f(z) \]  

(1.5)

Furthermore, for \( c > 0 \) the generalized Bernardi operator for meromorphic functions is defined as

\[ J_c f(z) = \frac{c}{z^{c+1}} \int_0^{\frac{1}{z}} f(t) dt \]  

(1.6)

From (1.6), we have

\[ (c+1) J_c f(z) + z J_c f(z) = cf(z), z \in D \]  

(1.7)
Many interesting classes of meromorphic functions have been recently studied by many authors [4, 5].

Using the operator $\theta^{\lambda}(\alpha)$, we define some classes of meromorphic functions as follow:

**Definition 1.1:** Let $f(z) \in M, \lambda > 0, 0 \leq \gamma < 1, z \in D$, then $f(z) \in MS(\alpha, \lambda, k, \gamma)$ if and only if

$$-\frac{z(\theta^\lambda(\alpha) f(z))'}{\theta^\lambda(\alpha) f(z)} \in P_1^\gamma$$

**Definition 1.2:** Let $f(z) \in M, \lambda > 0, 0 \leq \gamma < 1, z \in D$, then $f(z) \in MC(\alpha, \lambda, k, \gamma)$ if and only if

$$-\frac{z(\theta^\lambda(\alpha) f(z))'}{\theta^\lambda(\alpha) g(z)} \in P_1^\eta$$

We note that

$$f(z) \in MC(\alpha, \lambda, k, \gamma) \Rightarrow -zf'(z) \in MS(\alpha, \lambda, k, \gamma)$$

**Definition 1.3:** Let $f(z) \in M, \lambda > 0, 0 \leq \gamma, \eta < 1, z \in D$, then $f(z) \in MS^2(\alpha, \lambda, k, \gamma, \eta)$ if and only if there exists $g(z) \in MS(\alpha, \lambda, 2, \gamma)$ such that

$$-\frac{z(\theta^\lambda(\alpha) f(z))'}{\theta^\lambda(\alpha) g(z)} \in P_1^\eta$$

**Definition 1.4:** Let $f(z) \in M, \lambda > 0, 0 \leq \gamma, \eta < 1, z \in D$, then $f(z) \in MS^2(\alpha, \lambda, k, \gamma)$ if and only if there exists $g(z) \in MC(\alpha, \lambda, 2, \gamma)$ such that

$$-\frac{z(\theta^\lambda(\alpha) f(z))'}{\theta^\lambda(\alpha) g(z)} \in P_1^\eta$$

We choose a suitable function $\Phi(z)$ such that

$$-\frac{z(\theta^\lambda(\alpha) f(z))'}{\theta^\lambda(\alpha) g(z)} = H(z)$$ (3.1)

Using (1.5) and (3.1) we obtain

$$-\frac{z(\theta^\lambda(\alpha) f(z))'}{\theta^\lambda(\alpha) g(z)} = H(z) + \frac{zH'(z)}{-H(z) + (\lambda + 1)}$$ (3.2)

We choose a suitable function $\Phi(z)$ such that

$$\Phi(z) = \frac{1}{\lambda + 1} \left[ \frac{1}{z} + \sum_{j=1}^{\infty} \frac{1}{z^j} \right] + \frac{\lambda}{\lambda + 1} \left[ \frac{1}{z} + \sum_{j=1}^{\infty} \frac{1}{jz^j} \right]$$

then

$$H(z) = \Phi(z) = \frac{zH'(z)}{-H(z) + (\lambda + 1)}$$ (3.3)

Let

$$H(z) = \left( \frac{k}{4} + \frac{1}{2} \right) h_1(z) - \left( \frac{k}{4} - \frac{1}{2} \right) h_2(z)$$ (3.4)

From (3.2)-(3.4), we have

$$-\frac{z(\theta^\lambda(\alpha) f(z))'}{\theta^\lambda(\alpha) g(z)} = \left( \frac{k}{4} + \frac{1}{2} \right) h_1(z) + \frac{zH'(z)}{-H(z) + (\lambda + 1)}$$

**Lemma 1.1:** [6] Let $u = u_1 + iu_2$ and $v = v_1 + iv_2$ and let $\Psi(u, v)$ be a complex valued function satisfying the conditions:

(i) $\Psi(u, v)$ is continuous in $D \subseteq \mathbb{C}^2$.
(ii) $(1,0) \in D$ and $\Re \Psi(1,0) > 0$.
(iii) $\Re \Psi(iu_2, v_1) > 0$ whenever $(iu_2, v_1) \in D$ and $v_1 \leq -\frac{1}{2}(1 + u_2^2)$.

If $h(z)$ is a function analytic in E such that $(h(z), zh'(z)) \in D$ and $\Re \Psi(h(z), zh'(z)) > 0$ for $z \in \mathbb{E}$, then $\Re h(z) > 0$ in E.

**Lemma 1.2:** Let $h(z) \in P$, for $z \in \mathbb{E}$. Then

$$|zh'(z)| \leq \frac{2\Re e h(z)}{1 - |z|^2} [2]$$

$$\frac{1 - |z|^2}{1 + |z|^2} \leq \Re h(z) \leq \frac{1 + |z|^2}{1 - |z|^2} [9]$$

**MAIN RESULTS**

**Theorem 3.1:** Let $f(z) \in M$. Then

$MS(\alpha, \lambda + 1, k, \gamma) \subset MS(\alpha, \lambda, k, \gamma)$

Proof: First we prove

$MS(\alpha, \lambda + 1, k, \gamma) \subset MS(\alpha, \lambda, k, \gamma)$

Let $f(z) \in MS(\alpha, \lambda + 1, k, \gamma)$ and set

$$-\frac{z(\theta^\lambda(\alpha) f(z))'}{\theta^\lambda(\alpha) g(z)} = H(z)$$ (3.1)

Using (1.5) and (3.1) we obtain

$$-\frac{z(\theta^\lambda(\alpha) f(z))'}{\theta^\lambda(\alpha) g(z)} = H(z) + \frac{zH'(z)}{-H(z) + (\lambda + 1)}$$ (3.2)

We choose a suitable function $\Phi(z)$ such that

$$\Phi(z) = \frac{1}{\lambda + 1} \left[ \frac{1}{z} + \sum_{j=1}^{\infty} \frac{1}{z^j} \right] + \frac{\lambda}{\lambda + 1} \left[ \frac{1}{z} + \sum_{j=1}^{\infty} \frac{1}{jz^j} \right]$$

then

$$H(z) = \Phi(z) = \frac{zH'(z)}{-H(z) + (\lambda + 1)}$$ (3.3)

Let

$$H(z) = \left( \frac{k}{4} + \frac{1}{2} \right) h_1(z) - \left( \frac{k}{4} - \frac{1}{2} \right) h_2(z)$$ (3.4)

From (3.2)-(3.4), we have

$$-\frac{z(\theta^\lambda(\alpha) f(z))'}{\theta^\lambda(\alpha) g(z)} = \left( \frac{k}{4} + \frac{1}{2} \right) h_1(z) + \frac{zH'(z)}{-H(z) + (\lambda + 1)}$$
Since \( f(z) \in \text{MS}(\alpha, \lambda + 1, k, \gamma) \), therefore

\[
h_i(z) = \frac{z_{\alpha}(z)}{-h_i(z) + (\lambda + 1)} \in \mathbb{P}(\gamma) \text{ for } i = 1, 2; \ z \in \mathbb{E}.
\]

Let \( h_i(z) = \gamma_i + (1 - \gamma_i)p_i(z) \) for \( i = 1, 2 \). Then

\[
\frac{1}{1 - \gamma_i} \left[ (\gamma_i - 1) + (1 - \gamma_i)p_i(z) + \frac{(1 - \gamma_i)x_{\alpha}(z)}{-1 - \gamma_i}p_i(z) + (1 + \lambda - \gamma_i) \right] \in \mathbb{P}
\]

for \( i = 1, 2; \ z \in \mathbb{E} \).

We formulate a functional \( \Psi(u, v) \) by taking \( u = u_1 + iu_2 = p(z) \) and \( v = v_1 + iv_2 = z_{\alpha}(z) \), then

\[
\Psi(u, v) = \frac{1}{1 - \gamma_i} \left[ (\gamma_i - 1) + (1 - \gamma_i)u + \frac{(1 - \gamma_i)v}{(1 - \gamma_i)u + (1 + \lambda - \gamma_i)} \right]
\]

The first two conditions of Lemma 2.1 are obviously satisfied for \( \Psi(u, v) \) For the third condition, we proceed as follows:

\[
\text{Re} \Psi(iu, v_i) = \frac{A + Bu_i^2}{2C}
\]

where

\[
A = 2(\gamma_i - 1)(1 + \lambda - \gamma_i)^2 - (1 - \gamma_i)(1 + \lambda - \gamma_i) \]

\[
B = 2(\gamma_i - 1)(1 - \gamma_i)^2 - (1 - \gamma_i)(1 + \lambda - \gamma_i) \]

\[
C = \left[ (1 + \lambda - \gamma_i)^2 + (1 - \gamma_i)^2u_i^2 \right](1 - \gamma)
\]

Theorem 3.2: Let \( f(z) \in \mathbb{M} \). If \( f(z) \in \text{MS}(\alpha, \lambda, k, \gamma) \) then

\( Jf(z) \in \text{MS}(\alpha, \lambda, k, \gamma) \).

Proof: Let \( f(z) \in \text{MS}(\alpha, \lambda, k, \gamma) \) and set

\[
\frac{zJ_1(\theta_{\alpha}(\alpha) f(z))'}{J_1(\theta_{\alpha}(\alpha) f(z))} = H(z)
\]

where \( H(z) \) is analytic in \( \mathbb{E} \) and \( H(0) = 1 \). Using (1.7) and (3.6) we get

\[
\frac{z(\theta_{\alpha}(\alpha) f(z))'}{\theta_{\alpha}(\alpha) f(z)} = H(z) + \frac{zH'(z)}{-H(z) + (c + i)}
\]

Now using the same steps as in Theorem 2.1, we can prove that \( Jf(z) \in \text{MS}(\alpha, \lambda, k, \gamma) \), which completes the proof.

Theorem 3.3: If \( f(z) \) defined by (1.1) be in the class \( \text{MS}(\alpha, \lambda, k, \gamma) \) then

\[
|k_j| \leq \frac{(1 + \lambda)^2}{j!\gamma^2}, \ j \in \mathbb{N}
\]

Proof: Let \( f(z) \in \text{MS}(\alpha, \lambda, k, \gamma) \), then

\[
\frac{z(\theta_{\alpha}(\alpha) f(z))'}{\theta_{\alpha}(\alpha) f(z)} = H(z) \in \mathbb{P}(\gamma)
\]

where \( H(z) \) is analytic in \( \mathbb{E} \) and \( H(0) = 1 \). Let \( H(z) \) be of the form

\[
H(z) = 1 + \sum_{j=0}^{\infty} c_j z^j, \ z \in \mathbb{E}
\]

From (1.5), (3.7) and (3.8), we obtain

\[
\sum_{j=1}^{\infty} j\xi_p z^j = -\left[ \sum_{j=1}^{\infty} c_j z^j \right] - \left[ \sum_{j=1}^{\infty} \xi_p z^j \right] \left[ \sum_{j=1}^{\infty} c_j z^j \right], \ c_0 = 1
\]

By using Cauchy's product formula [3] for the power series, we obtain

\[
\sum_{j=1}^{\infty} j\xi_p z^j = -\left[ \sum_{j=1}^{\infty} c_j z^j \right] - \sum_{j=1}^{\infty} \xi_p z^j c_j + z^j
\]

Equating the coefficients of \( z^j \) on both sides, we have
\[
\sum_{\nu=0}^{m} c_{j\nu} \zeta_{j} = c_{j1} + \sum_{\nu=0}^{m} c_{j\nu} \zeta_{j\nu},
\]

Since \( H(z) \in P_{k}(\gamma) \), we have \(|c_{j}| \leq k(1-\gamma) \). This implies

\[
|b_{j}| \leq \frac{k(1-\gamma)}{j_{\zeta_{j}}} \left[ 1 + \sum_{\nu=0}^{m} |b_{j\nu}| \right], j \in \mathbb{N}
\]

By using induction on \( j \), we obtain

\[
|b_{j}| \leq \frac{(1+\zeta_{j})}{j_{\zeta_{j}}} \left[ k(1-\gamma) \right], j \in \mathbb{N}
\]

This completes the proof.

**Theorem 3.4:** Let \( f(z) \in M \). If \( f(z) \in MS(\alpha, \lambda, k, \gamma) \), then

\[
f(z) \in MS(\alpha, \lambda + 1, k, \gamma)
\]

for \(|z| < r_{0} \), where \( r_{0} \) is given by

\[
\tau_{j} = \frac{\lambda\left(2\lambda \gamma - 2\lambda - \gamma^{2}\right)}{\gamma + \sqrt{\lambda^{2} + \lambda(2-2\gamma + \gamma)}}
\]

**Proof:** Let \( F(z) = \psi^{*}f \). Let \( f(z) \in MS(\alpha, \lambda, k, \gamma) \), then

\[
\frac{z(\theta_{\zeta_{j}}(f(z)))^{*}}{\theta_{\zeta_{j}}(f(z))} = H(z)
\]

where \( H(z) \) is analytic in \( E \) and \( H(0) = 1 \). Using (1.5) and (3.10) we obtain

\[
\frac{z(\theta_{\zeta_{j}}(f(z)))^{*}}{\theta_{\zeta_{j}}(f(z))} = H(z) + \frac{zh(z)}{H(z) + (\lambda + 1)}
\]

Rearranging the terms, we have

\[
\frac{1}{1-\gamma} \left[ \frac{z(\theta_{\zeta_{j}}(f(z)))^{*}}{\theta_{\zeta_{j}}(f(z))} - \gamma \right] = \left( \frac{k}{4} + \frac{1}{2} \right) \left[ h_{1}(z) + \frac{\lambda_{h}(z)}{-((1-\gamma)(1+z^{2})+(\lambda + 1))} \right] - \left( \frac{k}{4} + \frac{1}{2} \right) \left[ h_{2}(z) + \frac{\lambda_{h}(z)}{-((1-\gamma)(1+z^{2})+(\lambda + 1))} \right]
\]

Now, for \( i = 1,2 \), we use Lemma 2.2, with \(|z| = r_{0} \), to have

\[
\Re \left[ \frac{zh_{i}(z)}{1-((1-\gamma)h_{i}(z) + \gamma) + (\lambda + 1)} \right] \geq \Re \left( \frac{2r}{1-r^{2}} - \lambda + 2r + (2-2\gamma + \gamma)r^{2} \right)
\]

The right side of above inequality is positive if \(|z| < r_{0} \), where \( r_{0} \) is given by (3.9).

**Theorem 3.5:** Let \( f(z) \in M \). Then

\[
MC(\alpha, \lambda + 1, k, \gamma) \subset MC(\alpha, \lambda, k, \gamma) \subset MC(\alpha + 1, \lambda, k, \gamma)
\]

**Proof:** Let \( f(z) \in MC(\alpha, \lambda + 1, k, \gamma) \), then

\[
-zf(z) \in MS(\alpha, \lambda + 1, k, \gamma)
\]

We prove

\[
f(z) \in MC(\alpha, \lambda, k, \gamma)
\]

\[
MC(\alpha, \lambda + 1, k, \gamma) \subset MC(\alpha, \lambda, k, \gamma)
\]

**Theorem 3.6:** Let \( f(z) \in M \). Then

\[
MS(\alpha, \lambda + 1, k, \eta) \subset MS(\alpha, \lambda, k, \eta)
\]

**Proof:** First we prove

\[
MS(\alpha, \lambda + 1, k, \eta) \subset MS(\alpha, \lambda, k, \eta)
\]
Let \( f(z) \in \text{MS}^r(\alpha, \lambda, k, \eta) \) and set
\[
-\frac{z(\theta_1(\alpha_1) f(z))'}{\theta_1(\alpha_1) g(z)} = H(z)
\]
(3.13)

Using (1.5) and (3.13) we obtain
\[
-\frac{z(\theta_2(\alpha_2) f(z))'}{\theta_2(\alpha_2) g(z)} = H(z) + \frac{z H'(z)}{-M(z) + (\lambda + 1)}
\]
where
\[
M(z) = -\frac{z(\theta_2(\alpha_2) g(z))'}{\theta_2(\alpha_2) g(z)}
\]

Since \( g(z) \in \text{MS}(\alpha, \lambda, 2, \gamma) \), therefore
\[
M(z) = \gamma + (1 - \gamma) m(z),\text{ where } m(z) \in \mathbb{P}
\]
(3.15)

Let
\[
H(z) = \left(\frac{k}{4} + \frac{1}{2}\right) h_i(z) - \left(\frac{k}{4} - \frac{1}{2}\right) h_2(z)
\]
(3.16)

From (3.14)-(3.16), we obtain
\[
-\frac{z(\theta_2(\alpha_2) f(z))'}{\theta_2(\alpha_2) g(z)} = \left(\frac{k}{4} + \frac{1}{2}\right) h_i(z) + \frac{zh_i'(z)}{(\lambda + 1 - \gamma - (1 - \gamma) m(z))}
\]
(3.17)

Since \( f(z) \in \text{MS}^r(\alpha, \lambda, 1, k, \eta) \), therefore (3.17) implies that
\[
h_i(z) + \frac{zh_i'(z)}{(\lambda + 1 - \gamma - (1 - \gamma) m(z))} \in \mathbb{P}(\eta) \text{ for } i = 1, 2; z \in \mathbb{E}
\]

Let \( h_i(z) = \eta_i + (1 - \eta_i) p_i(z) \) for \( i = 1, 2 \).

Then
\[
\left[ (\eta_i - \eta) + (1 - \eta_i) p_i(z) + \frac{(1 - \eta_i) z p_i'(z)}{(\lambda + 1 - \gamma) - (1 - \gamma) m(z)} \right] \in \mathbb{P}
\]
for \( i = 1, 2; z \in \mathbb{E} \).

We formulate a functional \( \Psi(u,v) \) by taking \( u = u_i + i u_2 = p(z) \) and \( v = v_i + i v_2 = z p_i(z) \), then
\[
\Psi(u,v) = (\eta_i - \eta) + (1 - \eta_i) u + \frac{(1 - \eta_i) v}{(\lambda + 1 - \gamma) - (1 - \gamma) m(z)}
\]

The first two conditions of Lemma 2.1 are obviously satisfied for \( \Psi(u,v) \). For the third condition, we proceed as follows:

\[
\Psi(iu, v_i) = (\eta_i - \eta) + (1 - \eta_i) iu + \frac{(1 - \eta_i) v_i}{(\lambda + 1 - \gamma) - (1 - \gamma) m(z)}
\]

Let \( m(z) = m_1 + im_2 \), after rationalizing the expression, we obtain
\[
\text{Re} \Psi(iu, v_i) = (\eta_i - \eta) + \frac{(1 - \eta_i) [(\lambda + 1 - \gamma) - (1 - \gamma) m_1]}{[(\lambda + 1 - \gamma) - (1 - \gamma) m_1] + [(1 - \gamma) m_2]}
\]

From \( v_i \leq -\frac{1}{2}(1 + u_i^2) \), we have
\[
\text{Re} \Psi(iu, v_i) \leq \frac{A + B u_i^2}{2C}
\]
where
\[
A = 2(\eta_i - \eta)^2 \left[ (\lambda + 1 - \gamma) - (1 - \gamma) m_1 \right]^2 + [(1 - \gamma) m_2] \]

\[
B = -(1 - \eta_i) [(\lambda + 1 - \gamma) - (1 - \gamma) m_1]
\]

\[
C = [(\lambda + 1 - \gamma) - (1 - \gamma) m_1]^2 + [(1 - \gamma) m_2]^2 \geq 0
\]

We note that \( \text{Re} \Psi(iu, v_i) \leq 0 \) if \( A \leq 0 \) and \( B \leq 0 \). From \( A \leq 0 \), we obtain
\[
2\eta_i \left[ (\lambda + 1 - \gamma) - (1 - \gamma) m_1 \right] + [(1 - \gamma) m_2] \]
\[
\eta_i = \frac{2}{2(\lambda + 1 - \gamma) - (1 - \gamma) m_1} \left[ (\lambda + 1 - \gamma) - (1 - \gamma) m_1 \right] + [(1 - \gamma) m_2] \]

By virtue of Lemma 2.1, we see that \( p_i \in \mathbb{P}, \text{ for } i = 1, 2 \) and \( z \in \mathbb{E} \). Hence \( h_i(z) \in P(\eta_i) \) which implies \( H(z) \in P(\eta_1) \) and consequently \( f(z) \in \text{MS}(\alpha, \lambda, k, \gamma, \eta) \).

REFERENCES


