On Monogeneity of Cyclic Quartic Fields of Prime Conductor

Mamoona Sultan, Toru Nakahara and Inayat Ali Shah

National University of Computer and Emerging Sciences [NUCES], Peshawar Campus, 160-Industrial Estate, Hayatabad, Khyber Pakhtunkhwa [K.P.K.], The Islamic Republic of Pakistan

Abstract: In this paper we want to give a new proof for non-monogenesis of any cyclic quartic fields K over the rationals Q of prime conductor congruent 1 modulo 4 except for the 5th cyclotomic field Q(exp(2πi/5)). This phenomenon was once proved using the Gauss sum attached to a quartic character by the second author. For Hasse's problem to determine whether an algebraic number field whose ring of integers has a power integral basis or not, Y. Motoda and the second author proved that infinitely many 2-elementary abelian fields with degree 8 have no power integral basis by solving Diophantine equations associated to seven quadratic subfields of an octic field L except for the 24th cyclotomic field Q(exp(2πi/24)). Our emphasis is applying a single simultaneous linear Diophantine equation to give a totally different and most succinct proof rather than the previous one for non-monogenesis of cyclic quartic fields K with prime conductor.

Key words: Non-monogeneity . cyclic quartic field . linear Diophantine equation . Hasse's conductor-discriminant theorem

INTRODUCTION

The characterization for an algebraic number field K whether the ring Z_K of integers of K has a power integral basis or not is known as Hasse's problem. For an algebraic number field tower Q⊂F⊂L over the field Q of rational numbers with the rings Z_F of integers in F and ring Z_L of integers in L, it is said that a field L is relatively monogenic in the relative field extension F/L of degree n if Z_L has a power integral basis \[ \theta \in \mathbb{Z}_F \] over Z_F, namely Z_L coincides with the Z_F module \[ \mathbb{Z}_F \cdot \theta \] of rank n. For the case of F = Q, we say that L is monogenic or that the ring Z_F has a power integral basis, if Z_L = Z⟦θ⟧ holds for an integer \( \theta \in \mathbb{Z} \).

Let \( \kappa_n \) be a cyclotomic field Q(\( \zeta_n \)) with a primitive root \( \zeta_n = \exp(2\pi i/n) \) for \( n \geq 3 \). \( \kappa_n \) and \( \kappa_n^* \) denote the maximal real subfield Q(\( \eta_n^* \)) with \( \eta_n^* = \zeta_n + \zeta_n^{-1} \) and a maximal imaginary subfield Q(\( \eta_n^* \)) with \( \eta_n^* = \zeta_n - \zeta_n^{-1} \) for \( n \geq 4 \) and 4|n respectively. Then \( \kappa_n \), \( \kappa_n^* \) and \( \kappa_n^* \) are monogenic, namely their rings \( \mathbb{Z}_{\kappa_n} \), \( \mathbb{Z}_{\kappa_n^*} \) and \( \mathbb{Z}_{\kappa_n^*} \) of integers have power integral bases \( \mathbb{Z}[\zeta_n] \), \( \mathbb{Z}[\eta_n^*] \) and \( \mathbb{Z}[\eta_n^*] \), respectively [1-3].

In the case of abelian quartic field K, we found many work to determine the monogeneity of K for cyclic extensions and biquadratic ones.

For an algebraic number field F, index Ind_F(\( \alpha \)) of an integer \( \alpha \) in F is defined by \( \sqrt{d_F(\alpha)} \) with the field discriminant \( d_F \) of F and the discriminant \( d_F(\alpha) \) of a number \( \alpha \) [3]. Certain biquadratic fields K whose minimum indices are greater than 1 and whose integral bases are explicitly determined by second author [4]. The field index Ind_F \( \frac{d_F}{\alpha} \) and minimum index \( m_F \) is defined by \( \gcd\{\text{Ind}_F(\alpha); \alpha \in \mathbb{Z}_F\} \) and \( m_F = \min \{\text{Ind}_F; \alpha \in \mathbb{Z}_F\} \).

M.-N. Gras and F. Tanoé contributed to Hasse's problem by providing a necessary and sufficient condition for the monogeneity of biquadratic field K = Q(\( \sqrt{m}, \sqrt{n} \)) using a diophantine equation of degree 4[5]. Their work is explored by Y. Motoda proving that there exist infinitely many monogenic biquadratic fields with some parameters [6]. The characterization of any 2-elementary abelian octifield

\[ F = Q(\sqrt{mn}, \sqrt{d}); \]

with

\[ mn = 3(\text{mod}4); d = 2(\text{mod}4); d > 0, \ell = 1(\text{mod}4) \]

where \( d/mn \) is square free, is proved to be non-monogenic using the seven linear equations of unit coefficients corresponding to seven quadratic subfield of F except for 24th cyclotomic field Q(exp(2πi/24)).
F = \sqrt{-1}, \sqrt{2}, \sqrt{-3}) where F coincide with the 24th cyclotomic field \( Q(\zeta_{24}) \) [7-9].

Let \( \zeta = \exp(2\pi i/n) \) and \( \tau(\chi) = \sum_{\chi(n)} \chi(n) \zeta^x \) be the Gauss sum attached to \( \chi \) of conductor \( n \), where \( G \) be the Galois group of a cyclotomic extension field \( k_{n}/Q \) and \( X = \langle \chi \rangle \) be the corresponding character group of \( G \) generated by \( \chi \). In 1982, the second author proved that the cyclic quartic field \( K \) with prime conductor \( p \) over \( Q \), is non-monogenic except for \( K = k_{5} \) by using Gauss sum \( \tau(\psi) \) attached to the quartic character \( \psi \) of conductor \( p \) as the main tool for the proof [10]. The aim of this paper is to give a new and simple proof of the non-monogeneity of the above cyclic quartic fields \( K \) of prime conductor \( p \) by making use of a single Diophantine equation with unit coefficients in the quadratic subfield \( Q(\sqrt{p}) \) of \( K \). It seems that this idea can be applied to determine the monogeneity of an abelian but non-cyclic octic extension field \( L \) including a cyclic quartic subfield with \( [L:Q] = 8 \).

**PRELIMINARY RESULTS**

We start with the following established lemmas and propositions available for our new proof.

**Lemma 2.1:** [11] (Hasse’s conductor- discriminant formula). Let \( K \) be an abelian number field associated to the group \( X \) of Dirichlet characters. Then the discriminant of \( K \) is given by

\[
d(K) = (-1)^{\frac{d}{2}} \prod_{\chi \in X} f_{\chi} \]

where \( f_{\chi} \) denotes conductor of \( \chi \) and \( r_{2} \) the number of the pair of complex conjugate field of \( K \).

**Lemma 2.2:** (Chain Theorem). For an extension field tower \( Q \subset k \subset K \), \( d_{k} \) and \( d_{k}/k \) be the discriminant of \( K \), \( k \) and the relative discriminant of \( K \) over \( k \), respectively. Then we have:

\[
d_{k} \cdot d_{k}/k = (-1)^{\frac{d_{k}}{2}} \prod_{\chi \in X} f_{\chi} \]

Here for a number \( \alpha \in K \) and an ideal \( \mathfrak{A} \) of \( K \), \( \alpha \subset \mathfrak{A} \) denote both side are equal to each other as ideals.

The next claim is a sufficient criterion for the non-monogeneity of a Galois extension field.

**Lemma 2.3:** [2] Let \( \ell \) be a prime number and \( F/Q \) be a Galois extension of degree \( n = ef \) with ramification index \( e \), the relative degree \( f \) and the decomposition degree \( g \) with respect to \( \ell \). If one of the following two conditions is satisfied, then the ring \( \mathbb{Z}_f \) of integers in \( F \) has no power integral basis, i.e., \( F \) is non-monogenic.

1. \( e\ell < n \) and \( f = 1 \);
2. \( e\ell \leq n + e - 1 \) and \( f \geq 2 \)

The following proposition can be proved directly by applying Lemma 2.1 for seven quadratic subfields of \( F \).

**Proposition 2.4:** [7] Let \( F \) be a 2-elementary abelian extension field \( Q(\sqrt{mn}, \sqrt{dn}, \sqrt{d}, \sqrt{m}, \sqrt{n}) \) with \( d, m, n, mn = 3, d = 2, dm, n, \ell = 1 \) (mod 4), \( d > 0 \) and \( \mathfrak{d} \) is square free. Then we have \( d_{F} = 2^{6}(\mathfrak{d} \mathfrak{m})^{4} \).

For the simplicity we restrict ourselves that \( d_{1}m_{1}n_{1} = 1 \). Let \( G = \langle \alpha, \beta, \gamma, \delta \rangle \) be the Galois group of \( F/Q \) with the identity \( \iota \). Then we have

**Remark 2.5:** It holds that

\[
\begin{align*}
k_{\alpha,\beta} &= Q(\sqrt{mn}) = k_{1}, \quad k_{\alpha,\gamma} = Q(\sqrt{dn}) = k_{2}, \\
k_{\alpha,\delta} &= Q(\sqrt{d}) = k_{3}, \quad k_{\alpha,\gamma} = Q(\sqrt{dm}) = k_{4}, \\
k_{\alpha,\delta} &= Q(\sqrt{m}) = k_{5}, \quad k_{\gamma,\delta} = Q(\sqrt{dn}) = k_{6}, \\
k_{\gamma,\delta} &= Q(\sqrt{m}) = k_{7}.
\end{align*}
\]

In [7], an infinite family of non-monogenic octic 2-elementary abelian extension fields was found. The proof of Proposition 2.6 developed in [7] is a basic tool to prove our theorem.

**Proposition 2.6:** [7] Let \( F \) be an octic field \( Q(\sqrt{mn}, \sqrt{dn}, \sqrt{d}) \) with \( dmn \) is square free and

\[
G = \langle \alpha, \beta, \gamma, \delta \rangle \subset \langle \alpha, \beta, \gamma, \delta \rangle.
\]

the Galois group of \( F/Q \). For \( \lambda, \mu \in G \) and \( k_{\lambda, \mu} \) be the fixed quadratic subfield and the fixed quartic one of \( F \) by the subgroups \( H_{\lambda, \mu} = \langle \lambda, \mu \rangle \) and \( H_{\lambda, \mu} = \langle \lambda, \mu \rangle \) of \( G \) respectively. If \( F \) is monogenic, then the following seven simultaneous equations corresponding to the quadratic subfields \( k_{\lambda, \mu} \subset (\lambda, \mu \in G) \) hold;

\[
d_{F/k_{\lambda, \mu}} E_{1} + d_{F/k_{\lambda, \mu}} E_{2} + d_{F/k_{\lambda, \mu}} E_{3} = 0, \quad E_{i} \in U_{k_{\lambda, \mu}} \quad (1 \leq j \leq 3)
\]
where \( d_{F/K} \) and \( U_{k,h} \) denote the relative discriminant \( \sqrt{d_{F/K}} \) of \( F/K \) and the unit group of an imaginary or a real quadratic field \( k_{h,m} \), respectively.

Since all the coefficients \( E_{ij} \) with \( i \neq 3 \) and \( 2E_{ij} \) with \( i = 3 \) in the above seven equations belong to \( \mathbb{Z}[1,D] \) for \( D = mn, dn, dm, \ldots, k_{h,m} \), it is enough to investigate a behavior on powers of units in the ring \( \mathbb{Z}[\sqrt{D}] \).

**Proposition 2.7:** [7] Let \( \varepsilon = t + u_1 \sqrt{D} = t + \sqrt{Du_1^2} \) be a unit in the ring \( \mathbb{Z}[\sqrt{D}] \) of a real quadratic field \( \mathbb{Q}(i) \) and \( \varepsilon' = t_u + u_1 \sqrt{D} = T + \sqrt{Du_1^2} \) and

\[
E = E_t + U_1 \sqrt{D} = T + \sqrt{Du_1^2},
\]

then we have \( U_1 = u_1/k_1 \)

**Lemma 2.8:** [7] Let \( E \) be a power \( \varepsilon' = t + u_1 \sqrt{D} = t + \sqrt{Du_1^2} > 1 \) of unit

\[
\varepsilon = t + u_1 \sqrt{D} = t + \sqrt{Du_1^2}
\]

in a quadratic field \( \mathbb{Q}(\sqrt{D}) \) and \( \sigma \neq t \) in \( \text{Gal}(\mathbb{Q}(\sqrt{D})/\mathbb{Q}) \). Let

\[
\begin{bmatrix}
E_1 & E_2 & E_3 \\
E_1' & E_2' & E_3'
\end{bmatrix}
\]

attached to the equations \( * \) by \( A_D \) and the rank of \( A_D \) by \( r_D \). Then we have a solution \( \{a,b,c\} \) of rational integers; \( a \pm b \pm c = 0 \) for \( r_D = 1 \)

\[
a/\pm U_1 = b/\pm U_1 = c/\pm U_1 = 1 \text{ for } r_D = 1
\]

with

\[
U_1 = U_{i,j} / U_1, \quad sg = j - k, \quad tg = k - j,
\]

\[
ug = i - j, (st, u) = 1, s + t + u = 0
\]

**NEW PROOF**

As an analogue of linear equations related to seven quadratic subfields of an octic 2-elementary abelian extension field, using a single simultaneous equation related to a unique quadratic subfield of a cyclic quartic field \( K \) with prime conductor \( \neq 5 \), we develop a new proof of non-monogeneity of the field \( K \).

**Theorem 3.1:** There does not exist any cyclic quartic subfield \( K \) of prime conductor \( p \) congruent to 1 modulo 4 whose ring \( \mathbb{Z}_K \) of integers has a power integral basis except for the 5th cyclotomic field

\[
k_5 = \mathbb{Q}(\exp(2\pi i/5))
\]

**Proof:** Consider a cyclotomic extension field \( k_5 = \mathbb{Q}(\zeta_5) \) of a prime conductor \( p \) congruent to 1 modulo 4 for a primitive \( p \)-th root of unity \( \zeta_p \). Let \( G \) be the Galois group \( \text{Gal}(k_5/Q) \) of \( k_5 \) over \( Q \) generated by an embedding \( \sigma : \zeta_p \mapsto \zeta_p^r \) with a primitive root \( r \) modulo \( p \). Let \( H_F \) denote the corresponding subgroup of \( G \) to a subfield \( F \) of \( k_p \). Let \( K \) and \( k \) be a cyclic quartic subfield \( \mathbb{Q}(\zeta) \) and a quadratic subfield \( \mathbb{Q}(\gamma) \) of \( k_p \) with Gauss periods

\[
\eta = \sum_{\rho \in H_k} \zeta_p^e \quad \text{of } \varphi(p)/4 \text{ terms and}
\]

\[
\gamma = \sum_{\rho \in H_k} \zeta_p^f \quad \text{of } \varphi(p)/2 \text{ terms, respectively. Here } H_k = \langle \sigma^f \rangle \text{ and } H_k = \langle \sigma^f \rangle \text{ are Galois subgroups corresponding to the subfields } K \text{ and } k \text{ respectively and } \varphi(.) \text{ denotes the Euler function.}
\]

Now for an abelian number field tower \( \mathbb{Q} \subset L \subset \mathbb{Q} \) with the Galois group \( G = (L/Q), \sigma \) and \( \sigma_{L/Q} \) denote the field different

\[
\{ \beta = \beta; \forall \beta \in Z_p, \forall \rho \in G(L/Q) \}
\]

of \( F \) and the relative field different

\[
\{ \gamma = \gamma; \forall \gamma \in Z_p, \forall \rho \in G(L/F) \}
\]

of \( L/F \), respectively. Then the field discriminant \( d_F \) and the relative field discriminant \( d_{L/F} \) are defined by \( N_L(d_F) \) and \( N_{L/F}(d_{L/F}) \), respectively, where for an ideal \( I \) of \( L \), \( N_{L/F}(I) \) means the ideal norm of \( I \) with respect to \( L/F \). Here we denote \( N_{L/0}(I) \) by \( N_L(I) \) for an ideal \( I \) in \( L \). Also the different \( \delta_{L/0}(\zeta_p) \) of an element \( \zeta_p \in Z_p \) is given by

\[
\delta_{L/0}(\zeta_p) = (\zeta_p - \zeta_p^e)(\zeta_p - \zeta_p^f) \cdots \cdots (\zeta_p - \zeta_p^{e+f+\cdots})
\]
and the different $d_k(ξ)$ of an element $ξ ∈ Z_k$ in the quartic subfield $K$ is given by

$$d_k(ξ) = (ξ - ξ') (ξ - ξ') (ξ - ξ') (ξ - ξ')$$  \hspace{1cm} (3.1)

Assume that $Z_k$ has a power integral basis generated by an integer $ξ ∈ Z_k$, namely $Ind_K(ξ) = 1$, here the group index $(Z_r : ζξ)$ of a sub module $Z[ξ]$ of $Z_r$ for $ξ ∈ Z_r$ coincides with the index $Ind_K(ξ)$ of an integer $ξ$. Taking the norm of both sides of (3.1)

$$N(ξ) = ξ' ξ' ξ' ξ'$$

and hence we obtain,

$$d_k = N_k (d_k(ξ)) = ξ' ξ' ξ' ξ'$$

and hence obtain,

$$d_k = (-1)^α A^2 B^2 C^2$$  \hspace{1cm} (3.2)

with

$$A = (ξ - ξ')(ξ - ξ')(ξ - ξ')(ξ - ξ')$$
$$B = (ξ - ξ')(ξ - ξ')(ξ - ξ')(ξ - ξ')$$
$$C = (ξ - ξ')(ξ - ξ')(ξ - ξ')(ξ - ξ')$$

and

$$A - B - C = 0$$  \hspace{1cm} (3.3)

Now for the abelian extension field tower $Q ⊂ k ⊂ K$, applying lemma 2.2, we obtain

$$d_{k} ≅ d_{k/k}$$  \hspace{1cm} (3.5)

Let $X_H$ denote the character group corresponding to a subgroup $H$ of $G(kp/Q)$ and hence

$$X_{G(kp/Q)} = <χ>, \hspace{0.5cm} X_{G(kp/Q)} = <δ>$$

for $δ = χ^{p(p)}/4$ and $X_{G(kp/Q)} = <λ>$ for $λ = χ^{p(p)}/2$ hold, where $χ$ denotes a primitive character of order $p$. By virtue of Hasse’s conductor-discriminant formula [12], the field discriminants are

$$d_k = \prod_{\rho \in F} p^ρ$$ and $d_k = \prod_{\rho \in F} p^ρ$

Here, $f_ρ$ denotes the conductor of a character $ρ ∈ X_{G(kp/Q)}$. So from (3.6), we have $p^ρ ≡ p^ρ d_{k/k}$ and hence $√p ≡ d_{k/k}$

Now for the Gauss period

$$η = η_p + η_p + η_p + η_p$$

$Z_k = Z[1, η, η', η'']$. Then for any $ξ ∈ Z_k$, we have $ξ = a_0 + a_1 η + a_2 η' + a_3 η''$ with $a_i ∈ Z(0 ≤ j ≤ 3)$. Thus

$$ξ' = a_0 + a_1 η' + a_2 η'' + a_3 η'''$$

For $(0 ≤ j ≤ 3)$ we have,

$$ξ - ξ' = a_0 (η - η') + a_1 (η' - η'') + a_2 (η'' - η''')$$

and by

$$η - η'' = (ξ_p - ξ_p) + (ξ_p - ξ_p) + (ξ_p - ξ_p)$$

it holds that $η'' - η'' = 0 (mod P)$ for each difference $η'' - η'' (0 ≤ r < s ≤ 3)$. Here $P$ denotes a ramified prime ideal $(1 - ζ_p)$ in $k_p$. Since $p$ is completely ramified in the field $k_p$, it holds that $p = Π(φ(p))$ and $Π(φ(p)) = P (1 ≤ j ≤ φ(p) - 1)$. Moreover there exists a prime ideal $\tilde{p}$ in $K$ such that $Π ⊂ Z_k = \tilde{p}$. It holds that $\tilde{p}^P$ with $p = \tilde{p}$ and $p ≡ p'$. Then $p = Π(φ(p))$ holds in $K$. By (3.5) we have

$$A = (ξ - ξ')(ξ - ξ')(ξ - ξ')(ξ - ξ')$$

Next we obtain the value of $C$ such that

$$\sqrt{p} ≡ A^2 B^2 C^2 (P)$$

Finally derive the value of $B$ using the equation (3.2), we have

$$(Q^{p(p)})^1 = (Q^{p(p)})^2 B^2 (Q^{p(p)})^{2^2}$$
namely

\[ \sqrt{p} \cong 2^{(p-1)/2} B \]

Now for the quadratic subfield \( k = \mathbb{Q}(\sqrt{p}) \) of \( k \), \( Z_k = \mathbb{Z}[1,\alpha] \) with \( \alpha = (1 + \sqrt{p})/2 \) let \( U_k \) denote the unit group \(<-1> \times <\varepsilon_0> \) of \( k \) with the normalized fundamental unit \( \varepsilon_0 > 1 \) of \( k \) where \( \varepsilon_0 = (a+b\sqrt{p})/2 \) with \( a, b \in \mathbb{Z} \) and \( 0 < b < a \). Let \( \rho \) be the real conjugate \( \neq i \) with respect to \( k/\mathbb{Q} \). For a prime conductor \( p \equiv 1(\mod 4) \), it is known that \( N(\varepsilon_0) = (a^2 - b^2p)/4 = -1 \) [12]. Thus from the linear relation (3.4) of three partial differents of a number \( \xi \) there exists three units \( \varepsilon_1, \varepsilon_2, \varepsilon_3 \in U_k \) such that

\[ \varepsilon_1 \sqrt{p} + \varepsilon_2 \sqrt{p} + \varepsilon_3 \sqrt{p} = 0 \] (3.6)

Then we obtain the simultaneous equations;

\[
\begin{align*}
\varepsilon_1 + \varepsilon_2 + \xi &= 0 \\
\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 &= 0
\end{align*}
\] (3.7)

If the equations with the coefficients \( \varepsilon_1, \varepsilon_2, \varepsilon_3, \xi \) have rank 1, then \( \varepsilon_1/\varepsilon_2 = \alpha \neq 1 \) holds. However by (3.7), it follows that \( \xi = 0 \) which is impossible. Therefore the rank of the system (3.7) of linear equations is equal to 2, namely \( \varepsilon_1, \varepsilon_2, \varepsilon_3 \neq 0 \) with \( 3\xi_j \in \mathbb{Q} \). Put \( \varepsilon_j = \xi_j^2 \).

Without loss of generality, we may assume that \( i_1 \leq i_2 \leq i_3 \) and \( i_1 \neq i_2 \) or \( i_2 \neq i_3 \). Then we obtain \( \varepsilon_0^\ell \pm \varepsilon_0^i + \varepsilon_0^i = 0 \) i.e. \( 1 \pm \varepsilon_0^i \pm \varepsilon_0^i = 0 \). Put \( j_1 = i_1, j_2 = i_2, j_3 = i_3 \), thus \( j_1 > 0 \) or \( j_2 > 0 \) holds. Hence with \( 0 \leq i_1 \leq i_2 \), it holds that \( 1 \pm \varepsilon_0^i \pm \varepsilon_0^i = 0 \). Put \( i_1 = \ell \) and \( i_2 = m \) with \( \ell < m \). Then the above equation is equivalent to

\[ 1 + \varepsilon_0^\ell - \varepsilon_0^m = 0 \] (3.8)

From (3.8), we have \( 1 + \varepsilon_0^\ell = \varepsilon_0^m \). Put

\[ \varepsilon_j = \varepsilon_0^j = (t_j + u_j \sqrt{p})/2 \]

Thus

\[ 1 + (t_1 + u_1 \sqrt{p})/2 = (2 + t_1 + u_1 \sqrt{p})/2 = (t_m + u_m \sqrt{p})/2 \]

Comparing the coefficient of \( \sqrt{p} \), we obtain \( u_1 = u_m \). Since \( \ell < m \), we have \( 1 < \ell + 1 < m \), so that \( \varepsilon_{i+1} \leq \varepsilon_m \) holds. This provides

\[ (t_{i+1} + u_{i+1} \sqrt{p})/2 \leq (t_m + u_m \sqrt{p})/2 \]

Then it holds that \( u_{i+1} \leq u_m \).

On the other hand, it follows that

\[ (t_{i+1} + u_{i+1} \sqrt{p})/2 = \varepsilon_{i+1} = \varepsilon_0 \]

Thus we obtain

\[ u_{i+1} = \varepsilon_0 = (t + u_1 \sqrt{p})/2 \]

Hence \( u_{i+1} \leq u_m = u_j \) holds, which lead to the inequality \( u_{i+1} \leq u_j \). However by [4],

\[ N_j(\varepsilon_j) = (a^2 - b^2p)/4 = -1 \]

Then it holds that \( a > 2 \) if \( p \neq 5 \), Then we obtain

\[ u_i = u_j < b / 2 + u_j \leq (t + u_j \sqrt{p})/2 = u_{i+1} \leq u_m = u_j \]

which deduces a contradiction because of \( p \neq 5 \).

Remark 3.1: In the case of 5th cyclotomic field \( k_5 \), the fundamental unit \( \varepsilon_0 \) of \( k \) is equal to \( (1 + \sqrt{5})/2 \). Then there exists a solution of (3.8) for \( \varepsilon_j = (6 + 2\sqrt{5})/4 \), namely,

\[ 1 + \varepsilon_0 - \varepsilon_0^j = 1 + (1 + \sqrt{5}/2 - (3 + \sqrt{5})/2 = 0 \]

In fact, \( Z_k \) has a power integral basis \( \mathbb{Z}[\xi, \zeta, \xi \zeta, \xi \zeta^2] \). Finally concerning the problem 6 in [13], we propose the next two related to Theorem 3.1.

Problem 3.1: Determine the monogeneity of an abelian but non-cyclic octic extension field \( L \) including a cyclic quartic subfield with \( [L:Q] = 8 \).

REFERENCES