An Efficient Numerical Method for the Solution of Coupled Two Dimensional Burger’s Equations

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Abstract: In this work, a semi-analytical method called Optimal Homotopy Asymptotic Method (OHAM) is implemented for the numerical solution of coupled two-dimensional Burger’s (CTDB) equations. It was revealed that only second- or the third-order OHAM solutions are sufficient to achieve the desired accuracy in comparison to the higher order solutions of the methods we have made comparison with. Some examples are given to show the effectiveness of the method.

Key words: Optimal homotopy asymptotic method · Coupled two-dimensional Burger’s equations · Solitary wave solution

INTRODUCTION

Most of the real world problems like those in physics, mechanics and biologics are governed by nonlinear partial differential equations (PDEs). For the last few decades, a great deal of research has been put in the direction of PDEs. Owing to complexity of such problems and the limitations of the available mathematical methods, very few of them have their exact solution and the researchers are forced to look for the numerical solutions. Various methods are available in the literature for the exact and numerical solution of these problems. These include Adomian’s decomposition method (ADM) [1-12], homotopy perturbation method (HPM) [13, 14, 16-20], variational iteration method (VIM) [21], Backlund transformation method [22, 23], homotopy analysis (HAM) [24, 25], and optimal homotopy asymptotic method [26-29].

In HPM, HAM and OHAM, the concept of homotopy from topology and conventional perturbation technique were merged to propose a general analytic procedure for the solution of nonlinear problems. Thus, these methods are independent of the existence of a small parameter in the problem at hand and thereby overcome the limitations of conventional perturbation technique. OHAM, however, is the most generalized form of the remaining two as it employs a more general auxiliary function \( H(q) \) in place of HPM’s \( -q \) and HAM’s \( h_q \). In this paper, we apply OHAM to solve the coupled system of two-dimensional Burger’s equations

\[
\frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} - \frac{1}{Re} \left( \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \right) = 0
\]

\[
\frac{\partial V}{\partial t} + U \frac{\partial V}{\partial x} + V \frac{\partial V}{\partial y} - \frac{1}{Re} \left( \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} \right) = 0
\]

where \( U(x, y, t) \), \( V(x, y, t) \) are the velocity components and \( Re \) is the Reynolds number. The initial conditions are:

\[
U(x, y, 0) = F_1(x, y), \quad V(x, y, 0) = F_2(x, y)
\]

The rest of the paper is organized as follows: In Section 2, the proposed method is described. OHAM solutions of the problem are given in Section 3, whereas Section 4 is devoted to the conclusion.

Method of Solution: In this section, we give an outline of the proposed method for a coupled system. For this, let us consider the following coupled equations

\[
L(U(x, y, t)) + f(x, y) + N(U(x, y, t)) = 0
\]

\[
L(V(x, y, t)) + f(x, y) + N(V(x, y, t)) = 0
\]

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where L and N are linear and nonlinear operators respectively, \( f, g \) are known functions, \( U, V \) are unknown functions and \( B \) is boundary operator. Thus OHAM is given by [26-29]

\[
(1-q)[L(\phi(\eta, \zeta, q)) + f(\eta, \zeta)] = H_1(q)[L(\phi(\eta, \zeta, q)) + f(\eta, \zeta) + N(\phi(\eta, \zeta, q))],
\]

\[
(1-q)[L(\psi(\eta, \zeta, q)) + g(\eta, \zeta)] = H_1(q)[L(\psi(\eta, \zeta, q)) + g(\eta, \zeta) + N(\psi(\eta, \zeta, q))]
\]

\[
B\left(U, V, \frac{\partial U}{\partial \eta}, \frac{\partial U}{\partial \zeta}, \frac{\partial V}{\partial \eta}, \frac{\partial V}{\partial \zeta}\right) = 0
\]

where \( q \in [0, 1] \) is an embedding operator, \( \Psi \) are unknown functions and \( H_1, H_2 \) are auxiliary functions such that

\[
H_1(q) = \begin{cases} 
\sum_{j=1}^{\infty} C_j q^j & \text{if } q \neq 0 \\
0 & \text{if } q = 0
\end{cases}
\]

\[
H_2(q) = \begin{cases} 
\sum_{j=1}^{\infty} D_j q^j & \text{if } q \neq 0 \\
0 & \text{if } q = 0
\end{cases}
\]

For our computational purpose

\[
H_1(q) = \begin{cases} 
\sum_{j=1}^{n} C_j q^j & \text{if } q \neq 0 \\
0 & \text{if } q = 0
\end{cases}
\]

\[
H_2(q) = \begin{cases} 
\sum_{j=1}^{n} D_j q^j & \text{if } q \neq 0 \\
0 & \text{if } q = 0
\end{cases}
\]

where \( C_j \)'s, \( D_j \)'s \((j = 1(1)n)\) are constants to be determined. By definition of homotopy

\[
\phi(\eta, \zeta, 0) = U_0(\eta, \zeta), \psi(\eta, \zeta, 0) = V_0(\eta, \zeta)
\]

\[
\phi(\eta, \zeta, 1) = U_1(\eta, \zeta), \psi(\eta, \zeta, 1) = V(\eta, \zeta)
\]

where \( U_0, V_0 \) are obtained from Eq. (5) and Eq. (6), for \( q = 0 \)

\[
L(U_0(\eta, \zeta)) + f(\eta, \zeta) = 0, L(V_0(\eta, \zeta)) + g(\eta, \zeta) = 0
\]

\[
B\left(U_0, V_0, \frac{\partial U_0}{\partial \eta}, \frac{\partial U_0}{\partial \zeta}, \frac{\partial V_0}{\partial \eta}, \frac{\partial V_0}{\partial \zeta}\right) = 0
\]

Now by Taylor’s series, we have
\( \phi(\eta, \zeta, q, C_j) = U_0(\eta, \zeta) + \sum_{k \geq 1} U_k(\eta, \zeta, C_j)q^k \)

\( \psi(\eta, \zeta, q, D_j) = V_0(\eta, \zeta) + \sum_{k \geq 1} V_k(\eta, \zeta, D_j)q^k \quad j = 1, 2, \ldots \) (14)

Where

\( U_k(\eta, \zeta, C_j) = \frac{1}{k!} \frac{\partial^k \phi(\eta, \zeta, q, C_j)}{\partial q^k} \bigg|_{q=0} \)

\( V_k(\eta, \zeta, D_j) = \frac{1}{k!} \frac{\partial^k \psi(\eta, \zeta, q, D_j)}{\partial q^k} \bigg|_{q=0} \) (15)

Using Eqs. (9)-(10) and Eqs. (14) in Eqs. (15)-(6), and equating like powers of \( q \)

\[
L(U_1(\eta, \zeta)) = C_1N_0(U_0(\eta, \zeta)), L(V_1(\eta, \zeta)) = D_1N_0(V_0(\eta, \zeta))
\]

\[
B \begin{bmatrix} U_1, V_1, \frac{\partial U_1}{\partial \eta}, \frac{\partial V_1}{\partial \zeta}, \frac{\partial U_1}{\partial \zeta}, \frac{\partial V_1}{\partial \eta} \end{bmatrix}
\]

\[
L(U_k(\eta, \zeta)) - L(U_{k-1}(\eta, \zeta)) = C_1N_0(U_0(\eta, \zeta)) + \sum_{j=1}^{k-1} C_j \left[ L_{k-j}(\eta, \zeta) + N_{k-j}(U_0(\eta, \zeta), U_1(\eta, \zeta), \ldots, U_{k-j}(\eta, \zeta)) \right]
\]

\[
L(V_k(\eta, \zeta)) - L(V_{k-1}(\eta, \zeta)) = D_1N_0(V_0(\eta, \zeta)) + \sum_{j=1}^{k-1} D_j \left[ L_{k-j}(\eta, \zeta) + N_{k-j}(V_0(\eta, \zeta), V_1(\eta, \zeta), \ldots, V_{k-j}(\eta, \zeta)) \right],
\]

\( k = 2, 3, \ldots \)

\[
B \begin{bmatrix} U_k, V_k, \frac{\partial U_k}{\partial \eta}, \frac{\partial U_k}{\partial \zeta}, \frac{\partial V_k}{\partial \eta}, \frac{\partial V_k}{\partial \zeta} \end{bmatrix} = 0
\]

\[
N_n(U_0(\eta, \zeta), U_1(\eta, \zeta), \ldots, U_n(\eta, \zeta)) = \frac{1}{n-1!} \frac{\partial^{(n-1)} N(\eta, \zeta, q)}{\partial q^{(n-1)}} \bigg|_{q=0}
\]

\[
N_n(V_0(\eta, \zeta), V_1(\eta, \zeta), \ldots, V_n(\eta, \zeta)) = \frac{1}{n-1!} \frac{\partial^{(n-1)} N(\eta, \zeta, q)}{\partial q^{(n-1)}} \bigg|_{q=0}
\] (19)

It should be noted that \( U_k, V_k \), for \( k = 1 \), are governed by Eqs. (12)-(18) which also involve their linear boundary conditions derived from the original problem and can be solved easily.

The convergence of the series (14) depend upon the auxiliary constants \( C_j, D_j, j \in \mathbb{N} \). Putting \( q = 1 \) in Eqs. (14), we get

\[
U(\eta, \zeta, C_j, D_j) = U_0(\eta, \zeta) + \sum_{k \geq 1} U_k(\eta, \zeta, C_j, D_j), \quad j = 1, 2, \ldots
\]

\[
V(\eta, \zeta, C_j, D_j) = V_0(\eta, \zeta) + \sum_{k \geq 1} V_k(\eta, \zeta, C_j, D_j), \quad j = 1, 2, \ldots
\] (20)
In actual calculation

\[ U^{(n)}(\eta, \xi, C_j, D_j) = U_0(\eta, \xi) + \sum_{k=1}^{n} U_k(\eta, \xi, C_j, D_j), \]

\[ V^{(n)}(\eta, \xi, C_j, D_j) = V_0(\eta, \xi) + \sum_{k=1}^{n} V_k(\eta, \xi, C_j, D_j), \quad j, k = 1, 2, \ldots, n. \]  

(21)

Substituting Eq. (21) into Eq. (3), we get the following residual

\[ R_1(\eta, \xi, C_j, D_j) = \frac{L(U^{(n)}(\eta, \xi, C_j, D_j)) + f(\eta, \xi) + N(U^{(n)}(\eta, \xi, C_j, D_j))}{1 + \frac{\partial^2 U}{\partial y^2}} \]

\[ R_2(\eta, \xi, C_j, D_j) = \frac{L(V^{(n)}(\eta, \xi, C_j, D_j)) + g(\eta, \xi) + N(V^{(n)}(\eta, \xi, C_j, D_j))}{1 + \frac{\partial^2 V}{\partial y^2}} \]  

(22)

When \( R(\eta, \xi, C_j, D_j) = R(\eta, \xi, C_j, D_j) = 0 \), then \( U^{(n)}(\eta, \xi, C_j, D_j), V^{(n)}(\eta, \xi, C_j, D_j) \) correspond to the exact solution.

However, when \( R(\eta, \xi, C_j, D_j) = R(\eta, \xi, C_j, D_j) \square= 0 \), we take a system of \( 2n \times 2n \) equations in unknowns \( C_j, D_j, j = 1(1)n \). For the purpose of comparison, we fix \( \eta = k, i = 1(1)n \) in Eq. (22) and equating to zero, we get

\[ R_{11}(k_1, 1, C_1, C_2, \ldots, C_n, D_1, D_2, \ldots, D_n) = R_{12}(k_2, 1, C_1, C_2, \ldots, C_n, D_1, D_2, \ldots, D_n) = \ldots = R_{1n}(k_n, 1, C_1, C_2, \ldots, C_n, D_1, D_2, \ldots, D_n) \]

\[ R_{21}(k_1, 1, C_1, C_2, \ldots, C_n, D_1, D_2, \ldots, D_n) = R_{22}(k_2, 1, C_1, C_2, \ldots, C_n, D_1, D_2, \ldots, D_n) = \ldots = R_{2n}(k_n, 1, C_1, C_2, \ldots, C_n, D_1, D_2, \ldots, D_n) \]

\[ \ldots \]

\[ R_{n1}(k_1, 1, C_1, C_2, \ldots, C_n, D_1, D_2, \ldots, D_n) = R_{n2}(k_2, 1, C_1, C_2, \ldots, C_n, D_1, D_2, \ldots, D_n) = \ldots = R_{nn}(k_n, 1, C_1, C_2, \ldots, C_n, D_1, D_2, \ldots, D_n) \]  

(23)

These equations can be easily solved for \( C_j, D_j \)'s to obtain the desired approximate solution.

**Implementation of the Method:** Now, we want to present OHAM solutions of Eq. (1) with

\[ F_1(x, y) = \frac{3}{4} - \frac{1}{4} \left( 1 + e^{\frac{-x+y}{8}} \right) \]

\[ F_2(x, y) = \frac{3}{4} + \frac{1}{4} \left( 1 + e^{\frac{-x+y}{8}} \right) \]  

(24)

In order to compare our solutions, we will use different values of Reynold numbers, namely \( \text{Re} = 50 \) and \( \text{Re} = 100 \). The operators of Eq. (3) are

\[ L(U(x, y, t)) = \frac{\partial U}{\partial t} + f(x, y) = 0, N(U(x, y, t)) = U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} - \frac{1}{\text{Re}} \left( \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \right) \]

\[ L(V(x, y, t)) = \frac{\partial V}{\partial t} + g(x, y) = 0, N(V(x, y, t)) = U \frac{\partial V}{\partial x} + V \frac{\partial V}{\partial y} - \frac{1}{\text{Re}} \left( \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} \right) \]  

(25)

Thus various order problems with their respective solutions are as under
Zero Order Problem:

\[
\begin{align*}
\frac{\partial U_0}{\partial t} &= 0, \quad \frac{\partial V_0}{\partial t} = 0 \\
\end{align*}
\]  
(26)

with initial conditions

\[
U_0 (x, y, 0) = F_1 (x, y), \quad V_0 (x, y, 0) = F_2 (x, y)
\]  
(27)

Solution of Zero Order Problem:

\[
U_0 (x, y, t) = \frac{2 + 3e^{\text{Re}(-x+y)}}{4 (1 + e^{\text{Re}(-x+y)})}, \quad V_0 (x, y, t) = \frac{4 + 3e^{\text{Re}(-x+y)}}{4 (1 + e^{\text{Re}(-x+y)})}
\]  
(28)

First Order Problem:

\[
\begin{align*}
\frac{1}{\text{Re}} \left[ C_1 \left( \frac{\partial^2 U_0}{\partial x^2} + \frac{\partial^2 U_0}{\partial y^2} \right) - \text{Re} \left\{ \frac{\partial U_0}{\partial t} + C_1 \frac{\partial U_0}{\partial t} + C_1 U_0 \frac{\partial U_0}{\partial x} - \frac{\partial U_1}{\partial t} + C_1 \frac{\partial U_0}{\partial y} - \frac{\partial U_0}{\partial y} V_0 \right\} \right] &= 0 \\
\frac{1}{\text{Re}} \left[ D_1 \left( \frac{\partial^2 V_0}{\partial x^2} + \frac{\partial^2 V_0}{\partial y^2} \right) - \text{Re} \left\{ \frac{\partial V_0}{\partial t} + D_1 \frac{\partial V_0}{\partial t} + D_1 U_0 \frac{\partial V_0}{\partial x} + D_1 V_0 \frac{\partial V_0}{\partial y} - \frac{\partial V_0}{\partial y} \right\} \right] &= 0
\end{align*}
\]  
(29)

Solution of First Order Problem:

\[
U_1 = \frac{1}{\text{Re}} \left( \frac{1}{128} e^{\text{Re}(-x+y)} \right)^2, \quad V_1 = -\frac{1}{\text{Re}} \left( \frac{128}{128} e^{\text{Re}(-x+y)} \right)^2
\]  
(30)

Second Order Problem:

\[
\begin{align*}
\frac{1}{\text{Re}} \left[ \text{Re} \left\{ \frac{\partial U_1}{\partial t} + \frac{\partial U_2}{\partial t} \right\} + C_2 \left[ \frac{\partial^2 U_0}{\partial x^2} + \frac{\partial^2 U_0}{\partial y^2} - \text{Re} \left\{ \frac{\partial U_0}{\partial t} + U_0 \frac{\partial U_0}{\partial x} + \frac{\partial U_0}{\partial y} \right\} \right] \right] &= 0 \\
\frac{1}{\text{Re}} \left[ C_1 \left( \frac{\partial^2 U_1}{\partial x^2} + \frac{\partial^2 U_1}{\partial y^2} + \text{Re} \left\{ \frac{\partial U_0}{\partial t} + U_0 \frac{\partial U_1}{\partial x} + \frac{\partial U_1}{\partial y} + \frac{\partial U_0}{\partial y} V_0 + \frac{\partial U_0}{\partial y} V_0 \right\} \right] \right] &= 0 \\
\frac{1}{\text{Re}} \left[ D_2 \left( \frac{\partial^2 V_0}{\partial x^2} + \frac{\partial^2 V_0}{\partial y^2} \right) - \text{Re} \left\{ \frac{\partial V_0}{\partial t} + D_2 V_0 \frac{\partial V_0}{\partial x} + \frac{\partial V_0}{\partial y} \right\} \right] &= 0 \\
\frac{1}{\text{Re}} \left[ D_1 \left( \frac{\partial^2 V_1}{\partial x^2} + \frac{\partial^2 V_1}{\partial y^2} \right) - \text{Re} \left\{ U_1 \frac{\partial V_0}{\partial x} + \frac{\partial V_1}{\partial y} V_0 + U_0 \frac{\partial V_0}{\partial x} + \frac{\partial V_0}{\partial y} + \frac{\partial V_0}{\partial y} \right\} \right] &= 0
\end{align*}
\]  
(31)

Solution of Second Order Problem:
Our second order OHAM solution can be obtained by using Eqs. (28), (30) and (32) in the following

\[
U^{(2)}(x, y, t, C_1, C_2, D_1, D_2) = U_0(x, y, t) + U_1(x, y, t) + U_2(x, y, t),
\]
\[
V^{(2)}(x, y, t, C_1, C_2, D_1, D_2) = V_0(x, y, t) + V_1(x, y, t) + V_2(x, y, t)
\]

Now, it only remains to determine the unknown constants \(C_j, D_j\) \((j = 1, 2)\).

For this, we use \(U^{(2)}, V^{(2)}\) for \(U, V\) in the left hand side expressions of Eqs.(1) to get the residuals

\[
R_1(x, y, t, C_1, C_2, D_1, D_2) = \frac{\partial U^{(2)}}{\partial t} + U^{(2)} \frac{\partial U^{(2)}}{\partial x} + V^{(2)} \frac{\partial U^{(2)}}{\partial y} - \frac{1}{\text{Re}} \left( \frac{\partial^2 U^{(2)}}{\partial x^2} + \frac{\partial^2 U^{(2)}}{\partial y^2} \right)
\]
\[
R_2(x, y, t, C_1, C_2, D_1, D_2) = \frac{\partial V^{(2)}}{\partial t} + U^{(2)} \frac{\partial V^{(2)}}{\partial x} + V^{(2)} \frac{\partial V^{(2)}}{\partial y} - \frac{1}{\text{Re}} \left( \frac{\partial^2 V^{(2)}}{\partial x^2} + \frac{\partial^2 V^{(2)}}{\partial y^2} \right)
\]

Both of the equations in Eq. (34) contain the unknown constants \(C_1, C_2, D_1, D_2\). Thus, to determine the values of these arbitrary constants, we evaluate and equate to zero the residuals at two different arguments of the domain of our interest [28]. The following system of four equations in four unknowns is obtained

\[
R_{11}(C_1, C_2, D_1, D_2) = R_{12}(C_1, C_2, D_1, D_2) = 0
\]
\[
R_{21}(C_1, C_2, D_1, D_2) = R_{22}(C_1, C_2, D_1, D_2) = 0
\]

This system was solved for \(x = \pm 0.01, y = 1\), to get values of OHAM constants \(C_j, D_j\) \((j = 1, 2)\) at different values of \(t\) and \(\text{Re} = 50, 100\). The values of OHAM constants at \(t = 0.5, \text{Re} = 50\) are

\[
C_1 = 0.02130915884261735, C_2 = -6.266561800701155,
\]
\[
D_1 = 0.0023340737284160423, D_2 = -4.97427282461488
\]

The solution so obtained is

\[
U^{(2)} = \frac{1}{2048\left(e^{25/4} + e^{25x/4}\right)^4} \left[ 1536e^{25x} + 1024e^{25x} + 2701.27e^{25x} \right. \left. + 3142.53e^{25x} + 2118.68e^{25x} \right]
\]
\[
V^{(2)} = \frac{1}{2048\left(e^{25/4} + e^{25x/4}\right)^4} \left[ 1536e^{25x} + 2048e^{25x} + 14727.7e^{25x} \right. \left. + 8643.84e^{25x} + 9667.84e^{25x} \right]
\]

\[\text{(36)}\]
The exact solution of the problem is

\[
U(x, y, t) = \frac{3}{4} \left( 1 + e^{\frac{1}{32} \frac{Ra(-4x+4y)}{1+e}} \right)
\]

\[
V(x, y, t) = \frac{3}{4} \left( 1 + e^{\frac{1}{32} \frac{Ra(-4x+4y)}{1+e}} \right)
\]

(37)

Fig. 1: Solution profile for \( V \) at \( t = 0.4, y = 1, Re = 50 \)

Fig. 2: Solution profiles for \( U \) and \( V \) at \( t = 0.4, y = 1, Re = 100 \)

Tables 1-4 show absolute error between exact and numerical solutions of order two for \( Re = 50 \) and \( Re = 100 \). It can be seen that our results are in good agreement with those obtained from five terms ADM [30] solutions.

Our OHAM solution profile for \( U, V \) at \( t = 0.4, y = 1 \) are depicted in Figs. 1-2. These results can be made even more accurate by increasing the order \( n \) of OHAM solutions.

CONCLUSION

In this work, OHAM was applied to obtain numerical solution of coupled nonlinear two-dimensional Burger’s equations (1). The method is fast converging and easy to implement as is evident from our second order solutions in comparison to ADM’s fifth order solutions. Furthermore, OHAM does not need any discretization in time or in space. Thus the solutions are not affected by
Table 1: Comparison of second order OHAM results for U with those obtained from ADM [30] (Re= 50)

<table>
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<th>t</th>
<th>x</th>
<th>ADM</th>
<th>OHAM</th>
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<td>3.92097E-04</td>
<td>1.29368E-04</td>
</tr>
<tr>
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<td>0.2</td>
<td>7.28716E-04</td>
<td>2.40298E-04</td>
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<td>0.4</td>
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Table 2: Comparison of second order OHAM results for V with those obtained from ADM [30] (Re= 50)

<table>
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<td>3.92097E-04</td>
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Table 3: Comparison of second order OHAM results for U with those obtained from ADM [30] (Re= 100)

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<td>1.66433E-03</td>
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<td>3.07805E-03</td>
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<td>5.64160E-03</td>
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<td>1.01664E-02</td>
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<td>3.64489E-03</td>
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<td>6.69102E-03</td>
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Table 4: Comparison of second order OHAM results for V with those obtained from ADM [30] (Re= 100)

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<th>ADM</th>
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</table>
computer round off errors and this leads to make use of lesser computer memory and time. The method can be easily extended to other nonlinear evaluation equations, with the aid of Mathematica (or Matlab, Maple etc.). In nutshell, OHAM is a better numerical tool for solving nonlinear evolution equations.

REFERENCES
