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Recurrence Relations for Single and Product Moments of Record Values from Chen Distribution and A Characterization

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Abstract: In this study we give some recurrence relations satisfied by single and product moments of upper record values from Chen distribution. Further a characterization of the Chen distribution based on conditional expectation of function of upper record values is presented.

Key words: Record . upper record . single moments . product moments . recurrence relations . Chen distribution . characterization . conditional expectation

INTRODUCTION

A random variable X is said to have the Chen distribution if its probability density function (pdf) is of the form

$$f(x)=\lambda \beta x^{\beta-1} e^{x^\beta} \exp(\lambda(1-e^{x^\beta})), \quad x, \lambda, \beta>0 \qquad (1.1)$$

and the cumulative distribution function (cdf) is given by

$$F(x)=1-\exp(\lambda(1-e^{x^\beta})); \quad x, \lambda, \beta>0 \qquad (1.2)$$

The Chen distribution given at (1.1) was introduced by Chen (2000). This is a two-parameter lifetime distribution with bathtub shape or increasing failure rate function. It is easy to see from (1.1) and (1.2) that for the Chen distribution

$$f(x)=\beta x^{\beta-1}[\lambda+\{-\log(1-F(x))\}][1-F(x)] \qquad (1.3)$$

Remark 1.1: By setting ß=1 in (1.1), Chen distribution reduces to

$$f(x)=\lambda e^{x} \exp(\lambda(1-e^{x})), \quad x, \lambda>0 \qquad (1.4)$$

Now let us consider Gompertz distribution with parameters a and ß. A random variable X is said to have Gompertz distribution if its probability density function (pdf) is of the form

$$f(x)=\beta e^{ax} \exp\left(\frac{\beta}{a}(1-e^{ax})\right), \quad x, a,\beta>0 \qquad (1.5)$$

Putting $\alpha = 1$ and $\beta = \lambda$ in (1.5) we get (1.4). Thus Gompertz distribution is a particular case of Chen distribution and results for Gompertz distribution can be derived as a special case from Chen distribution.

Suppose $\{X_n, n\geq 1\}$ is an infinite sequence of independent, identically distributed (iid) random variables with common cdf F(x) and pdf f(x), respectively. Let us assume that F(.) is continuous so that ties are not possible.

Let

$$Y_n = \max\{X_1, X_2, \ldots, X_n\}, n = 1, 2, \ldots$$

We say X_j is an upper record value of this sequence if $Y_j>Y_{j-1}$, $j\geq 2$. The indices at which the upper record values occur are given by the upper record times $\{U(n), n\geq 1\}$, where

$$U(n)=\min\{ j \mid j > U(n-1), X_j > X_{U(n-1)}, n>1\}$$

with U(1) = 1.

Then $X_{U(n)}$ and U(n) are the sequences of upper record values and upper record times, respectively.

Chandler (1952) introduced record values and record value times. Properties of record values of i.i.d. random variables have been extensively studied in the literature. Various developments on records and related topics have been reviewed by a number of authors including Glick (1978), Nevzorov (1987), Resnick (1987), Nagaraja (1988), Ahsanullah (1988, 1995), Arnold and Balakrishnan (1989) and Arnold, Balakrishnan and Nagaraja (1992, 1998).

In this paper, we established some recurrence relations satisfied by the single and product moments of upper record values from the Chen distribution in (1.1).

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A characterization of this distribution has also been obtained on using the conditional expectation of upper record values. Similar results for modified Weibull and Gompertz distribution have been derived by Sultan (2007) and Khan and Zia (2009). We shall denote

$$a_n^{(r)} = E(X_{U(n)}^r), \ r, n = 1,2,...$$
$$a_{m,n}^{(r,s)} = E(X_{U(m)}^r X_{U(n)}^s), \ 1 \le m \le n-1 \text{ and } r,s = 1,2,...$$
$$a_{m,n}^{(r,0)} = E(X_{U(n)}^r) = a_n^{(r)}, \ 1 \le m \le n-1 \text{ and } r = 1,2,...$$
$$a_{m,n}^{(0,s)} = E(X_{U(m)}^s) = a_m^{(s)}, \ 1 \le m \le n-1 \text{ and } s = 1,2,...$$

RELATIONS FOR SINGLE MOMENTS

Let $X_{U(1)} < X_{U(2)} < \ldots$ be the sequence of upper record values from (1.1). For convenience, we shall also take $X_{U(0)} = 0$. Then the pdf of $X_{U(n)}$, $n = 1,2,\ldots$, is given by

$$f_n(x) = \frac{1}{(n-1)!}\{-\log(1-F(x))\}^{n-1} f(x), \ -\infty<x<\infty \tag{2.1}$$

Theorem 2.1: For $n \ge 1$, $r = 0, 1, 2, \ldots$ and $ß > 0$,

$$\left(\frac{r}{ß}+1\right) a_n^{(r)} = (\lambda-n) a_n^{(r+ß)} - \lambda a_{n-1}^{(r+ß)} + n a_{n+1}^{(r+ß)} \tag{2.2}$$

Proof: We have from equations (1.3) and (2.1)

$$a_n^{(r)} = \frac{ß\lambda}{(n-1)!}\int_0^\infty x^{r+ß-1}\{-\log(1-F(x))\}^{n-1}[1-F(x)]dx$$
$$+\frac{ß}{(n-1)!}\int_0^\infty x^{r+ß-1}\{-\log(1-F(x))\}^{n}[1-F(x)]dx$$

Integrating by parts treating $x^{r+\beta-1}$ for integration and the rest of the integrand for differentiation and simplifying the so obtained relation, we immediately obtain the result of Theorem 2.1.

Remark 2.1: By setting ß=1 in (2.2) and rearranging, we get the recurrence relation for single moments of upper record values from Gompertz distribution with parameters a = 1 and ß = λ [see Khan and Zia (2009)].

RELATIONS FOR PRODUCT MOMENTS

We have the joint pdf of $X_{U(m)}$ and $X_{U(n)}$ $1 \le m < n$, as

$$f_{m,n}(x,y) = \frac{1}{(m-1)!(n-m-1)!}\{-\log(1-F(x))\}^{m-1}\{-\log(1-F(y)) +\log(1-F(x))\}^{n-m-1}.\frac{f(x)f(y)}{1-F(x)}, \ -\infty<x<y<\infty \tag{3.1}$$

Now, we derive some simple recurrence relations for the product moments of upper record values by using the relation (1.3).

Theorem 3.1: For $m \ge 1$, $r, s = 0,1,2,\ldots$ and $\beta > 0$

$$\left(\frac{r}{ß}+1\right) a_{m,m+1}^{(r,s)} = \lambda[a_m^{(r+ß+s)} - a_{m-1,m}^{(r+ß,s)}] + m[a_{m+1}^{(r+ß+s)} - a_{m,m+1}^{(r+ß,s)}] \tag{3.2}$$

and for $1 \le m \le n-2$, $r, s = 0,1,2,\ldots$ and $\beta > 0$

$$\left(\frac{r}{ß}+1\right) a_{m,n}^{(r,s)} = \lambda[a_{m,n-1}^{(r+ß,s)} - a_{m-1,n-1}^{(r+ß,s)}] + m[a_{m+1,n}^{(r+ß,s)} - a_{m,n}^{(r+ß,s)}] \tag{3.3}$$

Proof: From equation (3.1) for $1 \le m \le n-2$ and r, s = 0,1,2,… and on using equation (1.3), we get

$$a_{m,n}^{(r,s)} = \frac{1}{(m-1)!(n-m-1)!}\int_0^\infty y^s f(y) I(y)\, dy \tag{3.4}$$

Where

$$I(y) = ß\lambda\int_0^y x^{r+ß-1}\{-\log(1-F(x))\}^{m-1}\{-\log(1-F(y))+\log(1-F(x))\}^{n-m-1}dx$$
$$+ß\int_0^y x^{r+ß-1}\{-\log(1-F(x))\}^{m}\{-\log(1-F(y))+\log(1-F(x))\}^{n-m-1}dx$$

Integrating I(y) by parts treating $x^{r+\beta-1}$ for integration and the rest of the integrand for differentiation, we get

$$I(y) = \frac{ß\lambda(n-m-1)}{(r+ß)}\int_0^y x^{r+ß}\{-\log(1-F(x))\}^{m-1}\{-\log(1-F(y))+\log(1-F(x))\}^{n-m-2}\frac{f(x)}{[1-F(x)]}dx$$
$$-\frac{ß\lambda(m-1)}{(r+ß)}\int_0^y x^{r+ß}\{-\log(1-F(x))\}^{m-2}\{-\log(1-F(y))+\log(1-F(x))\}^{n-m-1}\frac{f(x)}{[1-F(x)]}dx$$
$$+\frac{ß(n-m-1)}{(r+ß)}\int_0^y x^{r+ß}\{-\log(1-F(x))\}^{m}\{-\log(1-F(y))+\log(1-F(x))\}^{n-m-2}\frac{f(x)}{[1-F(x)]}dx$$
$$-\frac{ßm}{(r+ß)}\int_0^y x^{r+ß}\{-\log(1-F(x))\}^{m-1}\{-\log(1-F(y))+\log(1-F(x))\}^{n-m-1}\frac{f(x)}{[1-F(x)]}dx$$

Substituting the above expression in (3.4) and simplifying the resulting equation, we get equation (3.3).

Proceeding in a similar manner for the case n = m+1, the recurrence relation given in (3.2) can easily be established.

Remark 3.1: By setting ß=1 in Theorem 3.1 and rearranging, we get the recurrence relation for product moments of upper record values from Gompertz distribution with parameters a=1 and ß = λ [see Khan and Zia (2009)].

Theorem 3.2: For m≥1, r, s = 0,1,2,… and

$$\left(\frac{s}{ß}+1\right)a_{m,m+1}^{(r,s)} = \lambda[a_{m,m+1}^{(r,s+ß)}-a_{m}^{(r+s+ß)}]+m[a_{m+1}^{(r+s+ß)}-a_{m+1,m+2}^{(r,s+ß)}]+[a_{m,m+2}^{(r,s+ß)}-a_{m,m+1}^{(r,s+ß)}] \tag{3.5}$$

and for 1≤m≤n-2, r, s = 0,1,2,… and β>0

$$\left(\frac{s}{ß}+1\right)a_{m,n}^{(r,s)} = \lambda[a_{m,n}^{(r,s+ß)}-a_{m,n-1}^{(r,s+ß)}]+m[a_{m+1,n}^{(r,s+ß)}-a_{m+1,n+1}^{(r,s+ß)}]+(n-m)[a_{m,n+1}^{(r,s+ß)}-a_{m,n}^{(r,s+ß)}] \tag{3.6}$$

Proof: From equation (3.1) for and r,s = 0,1,2,… and on using equation (1.3), we get

$$a_{m,n}^{(r,s)}=\frac{1}{(m-1)!(n-m-1)!}\int_0^\infty x^r\{-\log(1-F(x))\}^{m-1}\frac{f(x)}{1-F(x)}I(x)\,dx \tag{3.7}$$

where

$$I(x)=ß[\lambda+\{-\log(1-F(x))\}]\int_x^\infty y^{s+ß-1}\{-\log(1-F(y))+\log(1-F(x))\}^{n-m-1}[1-F(y)]dy+ß\int_x^\infty y^{s+ß-1}\{-\log(1-F(y))+\log(1-F(x))\}^{n-m}[1-F(y)]dy$$

Integrating I(x) by parts treating $y^{s+ß-1}$ for integration and the rest of the integrand for differentiation, we get

$$\begin{aligned}I(x) = {} & \frac{ß[\lambda+\{-\log(1-F(x))\}]}{(s+ß)}\int_x^\infty y^{s+ß}\{-\log(1-F(y))+\log(1-F(x))\}^{n-m-1}f(y)dy \\ & -\frac{ß[\lambda+\{-\log(1-F(x))\}](n-m-1)}{(s+ß)}\int_x^\infty y^{s+ß}\{-\log(1-F(y))+\log(1-F(x))\}^{n-m-2}f(y)dy \\ & +\frac{ß}{(s+ß)}\int_x^\infty y^{s+ß}\{-\log(1-F(y))+\log(1-F(x))\}^{n-m}f(y)dy-\frac{ß(n-m)}{(s+ß)}\int_x^\infty y^{s+ß}\{-\log(1-F(y))+\log(1-F(x))\}^{n-m-1}f(y)dy\end{aligned}$$

Substituting the above expression into equation (3.7) and simplifying the resulting equation, we get equation (3.6).

Proceeding in a similar manner for the case n = m+1, the recurrence relation given in (3.5) can easily be established.

CHARACTERIZATION

Let $X_{U(m)}$ and $X_{U(n)}$ be the m-th and n-th upper record values, then the conditional pdf of $X_{U(m)}$ given $X_{U(n)} = Y$, 1≤m<n

$$f_{m|n}(x|y) = \frac{(n-1)!}{(m-1)!(n-m-1)!}\{-\log(1-F(x))\}^{m-1}\{-\log(1-F(y))+\log(1-F(x))\}^{n-m-1}\frac{f(x)}{\{-\log(1-F(y))\}^{n-1}[1-F(x)]};\quad -\infty<x<y<\infty \tag{4.1}$$

and the conditional pdf of $X_{U(n)}$ given $X_{U(m)} = x$, 1≤m<n is

$$f_{n|m}(y|x)=\frac{1}{(n-m-1)!}\{-\log(1-F(y))+\log(1-F(x))\}^{n-m-1}\frac{f(y)}{[1-F(x)]};\quad -\infty<x<y<\infty \tag{4.2}$$

Theorem 4.1: Let X be an absolutely continuous rv with pdf f (x) and cdf F(x) on the support, then

$$F(x)=1-\exp(\lambda(1-e^{x^{\beta}})), \quad \lambda,\beta>0$$

if and only if

$$E\left[\exp(-\lambda e^{x^{\beta}_{U(r+1)}}) \mid x_{U(r)}=x\right]=\frac{\exp(-\lambda e^{x^{\beta}})}{2} \tag{4.3}$$

Proof: From equation (4.2), for n = r + 1 and m = r, we have

$$E\left[\exp(-\lambda e^{x^{\beta}_{U(r+1)}}) \mid x_{U(r)}=x\right]=\frac{1}{[1-F(x)]}\int_{x}^{\infty}\exp(-\lambda e^{y^{\beta}})f(y)\,dy \tag{4.4}$$

After using (1.1) and calculating the integral, we obtain (4.3).
To prove sufficient part, we have from (4.4)

$$\int_{x}^{\infty}\exp(-\lambda e^{y^{\beta}})f(y)dy=\frac{\exp(-\lambda e^{x^{\beta}})}{2}[1-F(x)]$$

Differentiating both the sides with respect to x and rearranging, we get

$$\frac{f(x)}{[1-F(x)]}=\lambda\beta x^{\beta-1}e^{x^{\beta}}$$

which leads to (4.2)

Remark 4.1: By setting ß=1 in Theorem 4.1, we get the Charecterization property Gompertz distribution with parameters a = 1 and ß = λ [see Khan and Zia (2009)].

DISCUSSION

Khan and Zia (2009) have obtained recurrence relations for single and product moments of upper record values from Gompertz distribution and a characterization. In this paper we have derived recurrence relations for single and product moments of upper record values from Chen distribution and a characterization. The results of Khan and Zia for Gompertz distribution can be obtained as a special case from Chen distribution.

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