On Bipolar Fuzzy Subgroups

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Abstract: In this paper, we have introduced the concept of Bipolar valued fuzzy subgroup, Bipolar valued fuzzy normal subgroups and investigate some associated results.

Key words:

INTRODUCTION

In 1965, in his pioneer paper L.A. Zadeh [9] introduced the concept of fuzzy set to handle the uncertainties in our daily life. After that many generalizations of fuzzy sets are presented, for example, interval valued fuzzy sets [10] and intuitionistic fuzzy sets [1]. Fuzzy sets are extremely useful to solve many problems in applied mathematics, information sciences and theory of automata [2]. The concept of fuzzy sets in Algebraic structures was first introduced by Rosenfeld in 1971 and he defined fuzzy subgroups [8].

In [5] K.M. Lee introduced the concept of Bipolar valued fuzzy sets. In [4] Jun and Song applied the notion of Bipolar valued fuzzy sets in BCH-algebra. Recently F. Nisar and others applied the notion of Bipolar valued fuzzy sets in BCI-algebra [7]. In this paper we used the notion of Bipolar valued fuzzy sets in group theory and develop some basic results.

Preliminaries: For undefined terms and notions we refer to [3, 6, 8].

Fuzzy subsets: A fuzzy subset of a non-empty set X is a function \( f: X \rightarrow [0,1] \). The set of all fuzzy subsets of a set X is denoted by FP(X). For \( f,g \in \text{FP}(X) \), we have the following operations on \( \text{FP}(X) \).

\[
(f \vee g)(x) = \max\{f(x),g(x)\} = f(x) \vee g(x)
\]
\[
(f \wedge g)(x) = \min\{f(x),g(x)\} = f(x) \wedge g(x)
\]

For all \( x \in X \).

Let \( A \) be a non-empty subset of a non-empty set X. Then characteristic function of \( A \) is a function \( C_A: X \rightarrow [0,1] \) defined by

\[
C_A(x) = \begin{cases} 
1 & \text{if } x \in A \\
0 & \text{if } x \notin A 
\end{cases}
\]

Obviously \( C_A \in \text{FP}(X) \).

Let \( x \) be an element of \( X \) and \( t \in (0,1] \). Then a fuzzy subset \( f \) of \( X \) of the form

\[
f(z) = \begin{cases} 
t & \text{if } z = x \\
0 & \text{if } z \neq x 
\end{cases}
\]

is called fuzzy point with value \( t \) and support \( x \) or fuzzy singleton subset of \( X \). It is denoted by \( t_x \).

Definition: Let \( G \) be a group and \( f \in \text{FP}(G) \). Then we have the following subsets of \( G \):

1. \( U(f,t) = \{ x \in G : f(x) \geq t \} \), \( t \in (0,1] \). It is called level subset of \( G \).
2. \( f^* = \{ x \in G : f(x) > 0 \} \). It is called support of \( f \).
3. \( f_e = \{ x \in G : f(x) = f(e) \} \). Where \( e \) is identity of \( G \).
4. \( \text{Ker}(f) = \{ x \in G : f(x) = 1 \} \). It is called Kernel of \( f \).

Definition: Let \( G \) be a group and \( f,g \in \text{FP}(X) \). Then, the fuzzy subsets \( f \circ g \) and \( f^{-1} \) of \( G \) are defined as:

\[
(f \circ g)(x) = \sup_{y \in X} \{ \min\{f(y),g(z)\} \}
\]
and

\[
f^{-1}(x) = f(x^{-1})
\]

where \( f \circ g \) is called product of \( f \) and \( g \) and \( f^{-1} \) is called inverse of \( f \).
Definition [6]: A fuzzy subset $f$ of a group $G$ is said to be a fuzzy subgroup of $G$ if for all $x, y \in G$

$$f(xy) \geq \min\{f(x), f(y)\}$$

or equivalently

$$f(x^{-1}) \geq f(x)$$

Definition [6]: A fuzzy subgroup $f$ of a group $G$ is said to be fuzzy normal subgroup of $G$, if for all $x, y \in G$

$$f(y^{-1}xy) \geq f(x)$$

Definition [6]: Let $f$ and $g$ be two fuzzy subgroups of a group $G$. Then, $f$ is said to be fuzzy conjugate of $g$ if for some $x \in G$, $f(y) = g(x^{-1}yx)$, for all $y \in G$.

Definition [6]: For a fuzzy subgroup $f$ of a group $G$, the normalizer of $f$ in $G$ is denoted and defined by

$$N(f) = \{y \in G: f(y^{-1}xy) \geq f(x)\}$$

for all $x \in G$.

Definition [6]: Let $f$ be a fuzzy subgroup of a group $G$. Then, fuzzy left coset of $f$ in $G$ determined by $x \in G$ is a fuzzy subset $f^l$ of $G$ defined by $f^l(y) = f(x^{-1}y)$.

Fuzzy right coset of $f$ in $G$ determined by $x \in G$ is a fuzzy subset $f^r$ of $G$ defined by $f^r(y) = f(xy^{-1})$.

Definition [6]: A fuzzy subgroup $f$ of a group $G$ is said to be abelian fuzzy subgroup of $G$ if and only if for all $x, y \in G$,

$$f(xy) = f(yx)$$

Remark [6]: Let $f$ be a fuzzy subgroup of a group $G$. Then

$$N(f) = \{y \in G: f(xy) = f(yx), \forall x \in G\}$$

Theorem [6]: Let $f$ be a fuzzy normal subgroup of a group $G$. Then, for all $x_1, x_2 \in G$,

$$f^l \circ f^r = f^r \circ f^l = f$$

Bipolar Valued Fuzzy Subgroups

Let $X$ be a non empty set. A Bipolar valued fuzzy subset $B$ of $X$ is an object having the form

$$B = \{(x, f^+(x), f^-(x)) : x \in X\}$$

where $f^+: X \to [0,1]$ and $f^- : X \to [-1,0]$ are mappings. The positive membership degree $f^+(x)$ denotes the satisfaction degree of an element $x$ to the property corresponding to a Bipolar valued fuzzy subset

$$B = \{(x, f^+(x), f^-(x)) : x \in X\}$$

and the negative membership degree $f^-(x)$ denotes the satisfaction degree of an element $x$ to some implicit counter property corresponding to a Bipolar valued fuzzy subset

$$B = \{(x, f^+(x), f^-(x)) : x \in X\}$$

If $f^+(x) \neq 0$ and $f^-(x) = 0$, it is the situation that $x$ has only positive satisfaction

$$B = \{(x, f^+(x), f^-(x)) : x \in X\}$$

If $f^+(x) = 0$ and $f^-(x) \neq 0$, it is the situation that $x$ does not satisfy the property of

$$B = \{(x, f^+(x), f^-(x)) : x \in X\}$$

but somewhat satisfies the counter property of

$$B = \{(x, f^+(x), f^-(x)) : x \in X\}$$

It is possible for an element $x$ to be such that $f^+(x) \neq 0$ and $f^-(x) \neq 0$ when the membership function of the property overlaps that of its counter property over some portion of $X$ [5]. For our convenience from now onward instead of writing

$$B = \{(x, f^+(x), f^-(x)) : x \in X\}$$

we will write $B = \langle f^+, f^- \rangle$.

Definition: Let $B_1 = \langle f^+, f^- \rangle$ and $B_2 = \langle g^+, g^- \rangle$ be two Bipolar valued fuzzy subsets of a group $G$. Then the Bipolar valued fuzzy subset $B_1 \lor B_2$ and $B_1 \land B_2$ of $X$ are defined as

$$B_1 \lor B_2 = \langle f^+ \lor g^+, f^- \lor g^- \rangle$$

and

$$B_1 \land B_2 = \langle f^+ \land g^+, f^- \land g^- \rangle.$$
Definition: Let $G$ be a group and $A \subseteq G$. Then Bipolar valued Characteristic function of $A$ is given by $C_A = <C^+_A, C^-_A>$, where

\[
C^+_A(x) = \begin{cases} 
1 & \text{if } x \in A \\
0 & \text{otherwise}
\end{cases}
\]

\[
C^-_A(x) = \begin{cases} 
-1 & \text{if } x \in A \\
0 & \text{otherwise}
\end{cases}
\]

Obviously Bipolar valued Characteristic function is Bipolar valued fuzzy subset of $G$.

Definition: Let $x$ be an element of a group $G$ and $t \in (0,1]$. Then a Bipolar valued fuzzy subset $f=<f^+, f^->$ of $G$ of the form

\[
f^+(z) = t \text{ if } z = x \]

\[
f^-(z) = -t \text{ if } z = x
\]

is called Bipolar valued fuzzy point with value $t$ and support $x$ or Bipolar valued fuzzy singleton subset of $G$. It is denoted by $x xxt t ,t +− =<>$.

Definition: Let $G$ be a group and $B_1=<f^+, f^->$ and $B_2=<g^+, g^->$ be two Bipolar valued fuzzy subsets of $G$. Then, the product of $B_1$ and $B_2$ is defined by

\[
B_1 \circ B_2 = <f^+ \circ g^+, f^- \circ g^->
\]

Where

\[
(f^+ \circ g^-)(x) = \sup_{z \in G} \left\{ \min(f^+(y), g^-(z)) \right\}
\]

and

\[
(f^- \circ g^+)(x) = \inf_{z \in G} \left\{ \max(f^-(y), g^+(z)) \right\}
\]

Definition: A Bipolar valued fuzzy subset $B=<f^+, f^->$ of a group $G$ is said to be Bipolar valued fuzzy subgroup of $G$ if

\[
f^+(y^{-1}xy) \geq f^+(x)
\]

and

\[
f^-(y^{-1}xy) \leq f^-(x)
\]

Example: Let $G = \mathbb{C}_4 = \{1, a, a^2, a^3\}$. Then $G$ is a group under usual multiplication of complex numbers. Let $B=<f^+, f^->$ be Bipolar valued fuzzy subset of $G$ such that

\[
f^+(1) = 0.75, f^+(a) = f^+(a^2) = 0.55, f^-(i) = f^-(i) = 0.25
\]

and

\[
f^-(1) = -0.83, f^-(1) = -0.63
\]

Then $B=<f^+, f^->$ is Bipolar valued fuzzy subgroup of $G$.

Theorem: Let $H$ be a non-empty subset of a group $G$. Then, $H \subseteq G$ if and only if its Bipolar valued characteristic function $C_H = <C^+_H, C^-_H>$ is Bipolar valued fuzzy subgroup of $G$.

Theorem: A Bipolar valued fuzzy subset $f=<f^+, f^->$ of a group $G$ is bipolar valued fuzzy subgroup of $G$ iff its level subset

\[
U(f, t) = \{x \in G : f^+(x) \geq t, f^-(x) \leq -t\}
\]

for all $t \in (0,1]$ is subgroup of $G$.

Theorem: Let $\{f_a = <f_a^+, f_a^-> : a \in I\}$ be a collection of Bipolar valued fuzzy subsets of a group $G$. Then, $\Lambda f_a$ is a Bipolar valued fuzzy subgroup of $G$.

Definition: A Bipolar valued fuzzy subgroup $f=<f^+, f^->$ of a group $G$ is said to be Bipolar valued fuzzy normal subgroup of $G$ if

\[
f^+(y^{-1}xy) \geq f^+(x) \text{ and } f^-(y^{-1}xy) \leq f^-(x)
\]

Example: Let $G = S_3 = \{1, a, a^2, b, ab, a^2b\}$ and $B=<f^+, f^->$ be a Bipolar valued fuzzy subset of $G$ such that

\[
f^+(1) = 0.85, f^+(a) = f^+(a^2) = 0.65
\]

\[
f^+(b) = f^+(ab) = f^+(a^2b) = 0.40
\]

and

\[
f^-(1) = -0.90, f^-(a) = f^-(a^2) = -0.59
\]

\[
f^-(b) = f^-(ab) = f^-(a^2b) = -0.30
\]

Then $B=<f^+, f^->$ is a Bipolar fuzzy normal subgroup of $G$.

Definition: Let $f=<f^+, f^->$ and $g=<g^+, g^->$ be two Bipolar fuzzy subsets of a group $G$. Then, $f= <f^+, f^->$ is said to be fuzzy conjugate of $g=<g^+, g^->$ if for some $x \in G$,
\[
f^*(y) = g^*(x^1y)
\]
\[
f(y) = g^*(x^1y)
\]
for all \(y \in G\).

**Definition:** For a Bipolar valued fuzzy subgroup \(f=<f^+, f^+>\) of a group \(G\), the Normalizer of \(f=<f^+, f^+>\) is denoted and defined by

\[
N(f) = \{y \in G: f^*(y^{-1}xy) \leq f^*(x)\}
\]
and \(f^*(y^{-1}xy) \leq f^*(x)\), for all \(x \in G\).

**Theorem:** Let \(f=<f^+, f^+>\) be a Bipolar fuzzy subgroup of a group \(G\). Then, \(N(f)\) is a subgroup of \(G\) if and only if \(N(f) = G\).

**Proof:**

1. Let \(y_1, y_2 \in N(f)\). Now for any \(x \in G\),

\[
f^*(y_1^{-1}xy_2) = f^*[(y_2y_1)x(y_1y_2)] = f^*[(y_2y_1)x] = f^*(y_1^{-1}xy_2)
\]

Now for any \(x \in G\),

\[
f^*(y_1^{-1}xy_2) = f^*(y_1^{-1}x) = f^*(y_1^{-1}x)
\]

and \(f^*(y_1^{-1}xy_2) \leq f^*(x)\),

\[
y_1y_2 \in N(f)
\]

**Definition:** Let \(f=<f^+, f^+>\) be a Bipolar valued fuzzy subgroup of a group \(G\). Then,

Bipolar valued fuzzy left coset of \(f=<f^+, f^+>\) in \(G\) determined by \(x \in G\) is a Bipolar valued fuzzy subset \(f^+ = f^+; f^+ >\) of \(G\) i.e. \(f^+ : G \to [0, 1]\) defined by \(f^+(y) = f^+(x^1y)\) and \(f^+ : G \to [-1, 0]\) defined by \(f^+(y) = f^+(x^1y)\).

Bipolar valued fuzzy right coset of \(f=<f^+, f^+>\) in \(G\) determined by \(x \in G\) is a Bipolar valued fuzzy subset \(f^+ = f^+; f^+ >\) of \(G\) i.e. \(f^+ : G \to [0, 1]\) defined by \(f^+(y) = f^+(yx^1)\) and \(f^+ : G \to [-1, 0]\) defined by \(f^+(y) = f^+(yx^1)\).

Conversely assume for all \(x, y \in G\), \(f^+ = f^+\) and \(f^+ = f^+\). To prove \(f=<f^+, f^+>\) is Bipolar valued fuzzy normal subgroup of \(G\).

Now for any \(x, y \in G\),

\[
f^+(yx) = f^+(yx) = f^+(yx) = f^+(yx)
\]

Similarly

\[
f^+(yx) \geq f^+(y)
\]

\[
f^+(yx) \leq f^+(y)
\]

\[
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\]
\[ f = \langle f^+, f^- \rangle \text{ is Bipolar valued fuzzy normal subgroup of } G. \]

**Definition:** A Bipolar valued fuzzy subgroup \( f = \langle f^+, f^- \rangle \) of a group \( G \) is said to be Bipolar valued fuzzy abelian subgroup of \( G \) if and only if for all \( x, y \in G \),

\[ f^+(xy) = f^+(yx) \text{ and } f^-(xy) = f^-(yx). \]

**Theorem:** For a Bipolar valued fuzzy subgroup \( f = \langle f^+, f^- \rangle \) of a group \( G \) the following are equivalent:

- \( f = \langle f^+, f^- \rangle \) is Bipolar valued fuzzy abelian subgroup of \( G \),
- \( f = \langle f^+, f^- \rangle \) is Bipolar valued fuzzy normal subgroup of \( G \).

**Remark:** Let \( f = \langle f^+, f^- \rangle \) be a Bipolar valued fuzzy subgroup of a group \( G \). Then,

\[ N(f) = \{ y \in G : f^+(xy) = f^+(yx) \} \]

and

\[ f^-(xy) = f^-(yx), \text{ for all } x \in G \]  

**Theorem:** Let \( f = \langle f^+, f^- \rangle \) be a Bipolar valued fuzzy subgroup of a group \( G \). Then, the cardinality of \( \{ f^0 = \langle f^{0+}, f^{0-} \rangle : u \in G \} \) is equal to index of \( N(f) \) in \( G \), where \( f^0 = \langle f^{0+}, f^{0-} \rangle \) is Bipolar valued conjugate fuzzy subgroup \( G \) defined by

\[ f^{0+}(x) = f^+(u^{-1}xu) \text{ and } f^{0-}(x) = f^-(u^{-1}xu), \text{ for all } x \in G. \]

**Proof:** To prove \( T = \{ f^0 = \langle f^{0+}, f^{0-} \rangle : u \in G \} \) and \( T = \{ uN(f) : u \in G \} \) are equivalent.

Define \( \phi : T \to T^* \) by \( \phi(f^0) = uN(f) \). For any \( u_1, u_2 \in G \),

\[ f^{u_1} = f^{u_2}; \]

\[ f^+(u_1^2xu_1) = f^+(u_1^2xu_1), \text{ for all } x \in G \]

\[ f^-(u_1^2xu_1) = f^-(u_1^2xu_1), \text{ for all } x \in G \]

Similarly, \( f^{u_1} = f^{u_2} \)

\[ f(u_1^2xu_1) = f(u_1^2xu_1), \text{ for all } x \in G \]

\[ f(u_1^2xu_1) = f(u_1^2xu_1), \text{ for all } x \in G \]

\[ u_1^2u_1 \in N(f), \text{ } u_1N(f) = u_2N(f) \]

so, \( \phi \) is bijective from \( T \) to \( T^* \). Hence the result.

**Remark:** With usual meanings, \( 1o f^0 = f^{0+} \), \( 1o f^0 = f^{0-} \) and \( f^0 o 1 = f^{0+} \), \( f^0 o 1 = f^{0-} \).

**Theorem:** Let \( f = \langle f^+, f^- \rangle \) be a Bipolar valued fuzzy normal subgroup of a group \( G \). Then, for all \( x_i, x_j \in G \),

(i) \( f^{x_i} \circ f^{-x_j} = f^{x_j} \circ f^{-x_i} = f^{-x_i} \circ f^{x_j} \).

(ii) \( f^{x_i} \circ f^{-x_j} = f^{x_j} \circ f^{-x_i} = f^{-x_j} \circ f^{x_i} \).

**Theorem:** Let \( G \) be a group and \( f = \langle f^+, f^- \rangle \) be a Bipolar valued fuzzy normal subgroup of \( G \). Then, the set \( G/f = \{ f^{(i)} : f^{(i)} > 0 \} \) is group. (It is called a factor group or quotient group).

**Proof:** For

\[ f^{(i)} = \langle f^{i+}, f^{i-} \rangle > \in G/f \]

define

\[ f^{i+} o f^{-i} = f^{i+} o f^{i-} o f^{i+} o f^{-i} >. \]

Then, \( G/f \) is closed.

Also, “\( o \)” is associative in \( G/f \). If \( e \) is identity in \( G \), then \( f^{(e)} = \langle f^{e+}, f^{e-} \rangle > \in G/f \) and for all

\[ f^{(i)} = \langle f^{i+}, f^{i-} \rangle > \in G/f \]

\[ f^{i+} o f^{i-} = f^{i+} \]

and

\[ f^{i+} o f^{i-} = f^{i+} \]

\[ f^{i+} o f^{i-} = f^{i+} \]

\[ f^{i+} o f^{i-} = f^{i+} \]

So

\[ f^{i+} o f^{i-} = f^{i+} \]

\[ f^{i+} o f^{i-} = f^{i+} \]

\[ f^{i+} o f^{i-} = f^{i+} \]

is identity in \( G/f \).

Also as for every

\[ f^{(i)} = \langle f^{i+}, f^{i-} \rangle > \in G/f \]

we have

\[ f^{(i)} = \langle f^{i+}, f^{i-} \rangle > \in G/f \]

such that
\[ f^+_i \circ f^-_i = f^+_i \circ f^-_i \circ f^+_i \circ f^-_i > \]
\[ = f^+_i \circ f^-_i \circ f^+_i \circ f^-_i > \]

and

\[ f^+_i \circ f^-_i = f^+_i \circ f^-_i \circ f^+_i \circ f^-_i > \]
\[ = f^+_i \circ f^-_i \circ f^+_i \circ f^-_i > \]

So for every \( f^+_i \in G/f \)

\[ f^+_i, f^-_i \in G/f \] and \( f^+_i, f^-_i \) acts as inverse of \( f^+_i \) in \( G/f \).

Hence \( G/f \) is group.

REFERENCES