Exact Solutions for Nonlinear Toda Lattice Difference Differential Equations

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Abstract: In this article, we put a direct new method to construct the rational solitary exact wave solutions for some nonlinear differential difference equations in mathematical physics which may be called the rational solitary wave difference method. We use the rational solitary wave functions method to construct some new exact rational solitary wave solutions for some nonlinear differential difference equations via the relativistic Toda lattice system and the discrete (2+1)-dimensional Toda lattice equation. The proposed method is more effective and powerful to obtain the exact solutions for nonlinear differential difference equations.

Key words: Rational solitary wave solutions • traveling wave solutions • the relativistic toda lattice system • the discrete (2+1)-dimensional toda lattice equation

INTRODUCTION

It is well known that the investigation of differential difference equations (DDEs) which describe many important phenomena and dynamical processes in many different fields, such as particle vibrations in lattices, currents in electrical networks, pulses in biological chains a many others and so on, has played an important role in the study of modern physics. Unlike difference equations which are fully discreted, DDEs are semi-discreted with some (or all) of their special variables discreted while time is usually kept continuous. DDEs also play an important role in numerical simulations of nonlinear partial differential equations (NLPDEs), queuing problems and discretization in solid state and quantum physics. Since the work of Fermi, Pasta and Ulam in the 1960s [1], DDEs have been the focus of many nonlinear studies. On the other hand, a considerable number of well-known analytic methods are successfully extended to nonlinear DDEs by researchers [2-17]. However, no method obeys the strength and the flexibility for finding all solutions to all types of nonlinear DDEs. Zhang et al. [18] and Aslan [19] used the \((G'/G)\)-expansion method to some physically important nonlinear DDEs. Qiong et al. [12] constructed the Jacobi elliptic solutions for nonlinear DDEs. Recently, Zhang et al. [20] and Gepreel [29-31] have used the Jacobi elliptic function method for constructing new and more general Jacobi elliptic function solutions of some nonlinear difference differential equations. The main objective of this paper, is to modify the rational solitary wave method which discussed by Xie [32] to solve the nonlinear differential difference equations instead of solving the nonlinear partial differential equations which may be called rational solitary wave difference method. We use this method to calculate the rational solitary wave solutions for some nonlinear DDEs in mathematical physics via the relativistic Toda lattice system and the discrete (2+1)-dimensional Toda lattice equation.

DESCRIPTION OF THE RATIONAL SOLITARY WAVE DIFFERENCE METHOD

In this section, we would like to outline on algorithm for using the rational solitary wave difference functions method to solve nonlinear DDEs. For a given nonlinear DDEs

\[
\Delta(u_{x_{r_1}}(x),...,u_{x_{r_k}}(x),u'_{x_{r_1}}(x),...,u'_{x_{r_k}}(x),...,u''_{x_{r_1}}(x),...,u''_{x_{r_k}}(x),...,v_{x_{r_1}}(x),...,v_{x_{r_k}}(x),...,v'_{x_{r_1}}(x),...,v'_{x_{r_k}}(x),...,v''_{x_{r_1}}(x),...,v''_{x_{r_k}}(x),...)=0
\]

where

\[
\Delta=(\Delta_1,...,\Delta_r), x=(x_1,x_2,...,x_n), n=(n_1,...,n_k)
\]

and \(g,m,Q,p_1,...,p_k\) are integers, \(u^{(l)},v^{(l)}\) denotes the set of all \(r^{th}\) order derivatives of \(u_1,v_1\) with respect \(x\).
The main steps of the algorithm for the rational solitary wave difference method to solve nonlinear DDEs are outlined as follows:

**Step 1:** We take the traveling wave solutions of the following form:

\[ u(x) = U(\xi), \quad v(x) = V(\xi), \ldots \]  

where

\[ \xi = \sum_{i=1}^{Q} d_i n_i + \sum_{j=1}^{m} c_j x_j + \xi_0 \]  

and \( d_i (i = 1, \ldots, Q) \), \( c_j (j = 1, \ldots, m) \), the phase \( \xi_0 \) are constants to be determined later. The transformations (2) is reduced Eqs. (1) to the following nonlinear ordinary differential difference equations

\[ \Omega(U(\xi_{m,n}), \ldots, U'(\xi_{m,n}), U(\xi_{m,n+1}), \ldots, U'(\xi_{m,n+1}), \ldots, V(\xi_{n,m}), \ldots, V'(\xi_{n,m}), \ldots, V'(\xi_{n+1,m}), \ldots) = 0 \]  

where \( \Omega = (\Omega_1, \ldots, \Omega_g) \).

**Step 2:** We suppose the rational solitary wave series expansion solutions of Eqs (4) in the following form:

\[ U(\xi_n) = \sum_{i=0}^{N} a_{i}\left[g(\xi_n)\right]^{i} f(\xi_n), \quad V(\xi_n) = \sum_{i=0}^{N} \beta_{i}\left[g(\xi_n)\right]^{i} f(\xi_n), \ldots \]  

with

\[ f(\xi_n) = \frac{1}{\text{Atanh}(\xi_n) + \text{Bsech}(\xi_n)}, \quad g(\xi_n) = \frac{\text{sech}(\xi_n)}{\text{Atanh}(\xi_n) + \text{Bsech}(\xi_n)} \]  

which satisfy

\[ f'(\xi_n) = \frac{1}{\text{Atanh}(\xi_n) + \text{Bsech}(\xi_n)}, \quad g'(\xi_n) = \frac{\text{sec}(\xi_n)}{\text{Atanh}(\xi_n) + \text{Bsech}(\xi_n)} \]  

Equations (7) and (9) can be written into unified form.
\[
f' (\xi_n) = -A g' (\xi_n) + \delta \frac{B g (\xi_n)}{A} [1 - B g (\xi_n)], \quad g' (\xi_n) = -A f (\xi_n) g (\xi_n), f' (\xi_n) = g^2 (\xi_n) + \delta \frac{1}{A[1 - B g (\xi_n)]} f',
\]

\[
\gamma (\xi_n \pm d) = \frac{A f (d) f (\xi_n) \pm \delta [1 - B g (\xi_n)][1 - B g (d)]}{A f (d) [1 - B g (\xi_n)] \pm A f (\xi_n) [1 - B g (d)] + B g (\xi_n) g (d)},
\]

\[
g (\xi_n \pm d) = \frac{g (\xi_n) g (\xi_n)}{f (\xi_n) [1 - B g (\xi_n)] \pm f (\xi_n) (1 - B g (d)) + B g (\xi_n) g (d)}.
\]

**Step 4:** Determine the degree N.L,… of Eqs. (4) by balancing the nonlinear term(s) and the highest order derivatives of U(\(\xi_n\)), V(\(\xi_n\)),… in Eq. (4). It should be noted that the leading terms U(\(\xi_{n+2p}\)), V(\(\xi_{n+2p}\))…p≠0 will not affect the balance because we are interested in balancing the terms of \(f (\xi_n)\) and \(g (\xi_n)\).

**Step 5:** Substituting Eq. (5) and (10) the given values of K,L,… into Eq.(4). Cleaning the denominator and collecting all terms with the same degree of \(f (\xi_n)\) and \(g (\xi_n)\) together, the left hand side of Eq. (3) is converted into a polynomial in \(f (\xi_n)\) and \(g (\xi_n)\). Setting each coefficient \(f' (\xi_n), g' (\xi_n)\) to be zero, we derive a set of algebraic equations for \(a_i, \alpha_i, b_j, \beta_j, C_i, A, B\).

**Step 6:** Solving the over determined system of nonlinear algebraic equations by using Maple or Mathematica. We end up with explicit expressions for \(a_i, \alpha_i, b_j, \beta_j, C_i, A, B\).

**Step 7:** Substituting \(a_i, \alpha_i, b_j, \beta_j, C_i, A, B\) into Eq. (5) along with (6) and (8), we can finally obtain the rational solitary wave solutions for nonlinear difference differential equations (1).

**APPLICATIONS**

In this section, we apply the proposed rational solitary wave difference method to construct the traveling wave solutions for some nonlinear DDEs via the relativistic Toda lattice system and the discrete (2+1)-dimensional Toda lattice equation which are very important in the mathematical physics and have been paid attention by many researchers.

**Example 1. The relativistic Toda lattice system:** In this section, we study the relativistic Toda lattice system which takes the following form [33]

\[
\frac{d u_n (t)}{d t} = (1 + \alpha u_n) (v_n - v_{n-1}),
\]

\[
\frac{d v_n (t)}{d t} = v_n (u_{n+1} - u_n + \alpha v_{n+1} - \alpha v_{n-1})
\]

(20)

where \(\alpha\) is an arbitrary constant. If, we take the transformation

\[
u_n = \frac{-1}{\alpha} u_n - \frac{1}{\alpha^n}
\]

(21)

The transformation (21) reduced the relativistic Toda lattice system (20) into the following differential difference equation

\[
\frac{d u_n (t)}{d t} = (u_n + \frac{1}{\alpha}) (u_{n-1} - u_n)
\]

(22)

According to the above steps, to seek traveling wave solutions of Eq. (22), we construct the traveling wave transformation

\[
\xi_n = d n + c t + \xi_0
\]

(23)

where \(\xi\) and \(\xi_0\) are constants. The transformation (23) permits us converting Eq. (22) into the following form:

\[
c U' (\xi_n) = (U_n + \frac{1}{\alpha}) [U (\xi_n - d) - U (\xi_n)]
\]

(24)

where \(\gamma = d / d\xi_n\). Considering the homogeneous balance between the highest order derivative and the nonlinear term in (24), we get \(N = 1\). Thus the solution of Eq. (24) has the following form:

\[
U (\xi_n) = a_0 + a_1 f (\xi_n) + b g (\xi_n)
\]

(25)

where \(a_0, a_1\) and \(b\) are constants to be determined later.

With the aid of Maple, substituting Eq.(25) and Eqs.(10) into Eq.(24) and collecting all terms with the same power in

\[
f' (\xi_n), g' (\xi_n) (i = 0,1, j = 0,1,2,\ldots)
\]

Setting the coefficients of these terms

\[
f' (\xi_n), g' (\xi_n) (i = 0,1, j = 0,1,2,\ldots)
\]

to be zero yields a set of algebraic equations which have the following solutions:

When \(\delta = 1\)

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Case 1

\[ a_0 = \frac{1}{\alpha(e^\alpha - 1)} \left[ \pm \frac{\alpha a_e(e^\alpha + 1)}{\sqrt{A^2 + B^2}} + (1 - e^\alpha) \right], \quad a_i = \frac{\pm \alpha a_e}{\sqrt{A^2 + B^2}}, \quad c_i = \frac{\pm 2a_e}{\sqrt{A^2 + B^2}} \] (26)

where \( A, B, a_2, \alpha \) are arbitrary constants. In this case the rational solitary wave solution for the nonlinear relativistic Toda lattice system have the following form:

\[ U(\xi_n) = \frac{1}{\alpha(e^\alpha - 1)} \left[ \pm \frac{\alpha a_e(e^\alpha + 1)}{\sqrt{A^2 + B^2}} + (1 - e^\alpha) \right] \pm \frac{Aa_e}{\sqrt{A^2 + B^2} \left[ \text{Atanh}(\xi_n) + B\text{sech}(\xi_n) \right]} + \frac{a_i \text{sech}(\xi_n)}{\left[ \text{Atanh}(\xi_n) + B\text{sech}(\xi_n) \right]} \] (27)

where

\[ \xi_n = d_n \pm \frac{2a_e}{\sqrt{A^2 + B^2}} t + \xi_0 \] (28)

When \( \delta = -1 \)

Case 2

\[ a_0 = \frac{1}{\alpha \sin(d)} \left[ \pm \frac{\alpha a_e(\cos d + 1)}{\sqrt{A^2 - B^2}} + \sin(d) \right], \quad a_i = \frac{\pm \alpha a_e}{\sqrt{A^2 - B^2}}, \quad c_i = \frac{\pm 2a_e}{\sqrt{A^2 - B^2}} \] (29)

In this case the rational solitary wave solution for the nonlinear relativistic Toda lattice system have the following form:

\[ U(\xi_n) = \frac{1}{\alpha \sin(d)} \left[ \pm \frac{\alpha a_e(\cos d + 1)}{\sqrt{A^2 - B^2}} - \sin(d) \right] \pm \frac{Aa_e}{\sqrt{A^2 - B^2} \left[ \text{Atan}(\xi_n) + B\text{sec}(\xi_n) \right]} + \frac{a_i \text{sec}(\xi_n)}{\left[ \text{Atan}(\xi_n) + B\text{sec}(\xi_n) \right]} \] (30)

where

\[ \xi_n = d_n \pm \frac{2a_e}{\sqrt{A^2 - B^2}} t + \xi_0 \] (31)

Example 2: The discrete (2+1)-dimensional Toda lattice equations

In this section, we apply the method developed to the following discrete (2+1)-dimensional Toda lattice equations

\[ \frac{\partial y_n}{\partial x} = e^{\gamma_n - \gamma_x - \gamma_n} + e^{\gamma_n - \gamma_x} \] (32)

where \( y_n = y_n(x,t) \) is the displacement from equilibrium of unit mass under an exponentially decaying interaction force nearest neighbors. The transformation \( \frac{\partial u_n}{\partial t} = e^{\gamma_n - \gamma_x - \gamma_n} - 1 \), reduced the discrete (2+1)-dimensional Toda lattice equation (32) to the following form:

\[ \frac{\partial u_n}{\partial x} = \left( \frac{\partial u_n}{\partial t} + u_{n-1} - 2u_n + u_{n+1} \right) \] (33)

Let

\[ u_j(x,t) = U(\xi_n), \quad \xi_n = d_n + c_1 x + c_2 t + \xi_0 \]

then Eq. (33) can be written in the form:

\[ c_1 c_2 U'(\xi_n) = (\xi U(\xi_n) + 1)U(\xi_n - d) - 2U(\xi_n) + U(\xi_n + d) \] (34)

According the homogeneous balance procedure, we get the solutions of Eq.(34) in the form:

\[ U(\xi_n) = a_0 + a_1 f(\xi_n) + b g(\xi_n) \] (35)

where \( a_0, a_1, b_1 \) are constants to be determined later. With the aid of Maple, substituting Eqs.(35) and (10) into Eq. (24) and collecting all terms with the same power in \( f'(\xi_n), g'(\xi_n) (i = 0,1,j = 0,1,2,...) \). Setting the coefficients of these terms \( f'(\xi_n), g'(\xi_n) (i = 0,1,j = 0,1,2,...) \) to be zero yields a set of algebraic equations which have the following solutions:

When \( \delta = 1 \)

Case 3

\[ a_1 = \frac{A[C\cosh(d) - 1]}{C_2}, \quad a_2 = \frac{\pm \sqrt{A^2 + B^2} \left[ \cosh(d) - 1 \right]}{C_2}, \quad c_i = \frac{2[C\cosh(d) - 1]}{C_2} \] (36)

where \( A,B,C_2,a_0,d \) are arbitrary constants. In this case the rational solitary wave solution for the discrete
(2+1)-dimensional Toda lattice equation have the following form:

\[
U(\xi_n) = a_0 + \frac{A[\cosh(d)-1]}{C[\text{Atanh}(\xi_n) + \text{Bsech}(\xi_n)]} \pm \frac{\sqrt{A^2 + B^2}[\cosh(d)-1]\text{sech}(\xi_n)}}{C[\text{Atanh}(\xi_n) + \text{Bsech}(\xi_n)]}
\]

(37)

where

\[
\xi_n = d n + \frac{2x[\cosh(d)-1]}{C} + C_2 t + \xi_0
\]

When \(\delta = -1\)

**Case 4**

\[
a_1 = -\frac{A[\cos(d)-1]}{C^2}, \quad a_2 = \frac{\sqrt{A^2 - B^2}[\cos(d)-1]}{C^2},
\]

(38)

where \(a_0, A, B, C_2\) are arbitrary constants. In this case the rational solitary wave solution for the discrete (2+1)-dimensional Toda lattice equation have the following form:

\[
U(\xi_n) = a_0 - \frac{A[\cos(d)-1]}{C[\text{Atan}(\xi_n) + \text{Bsec}(\xi_n)]} \pm \frac{\sqrt{A^2 - B^2}[\cos(d)-1]\text{sec}(\xi_n)}}{C[\text{Atan}(\xi_n) + \text{Bsec}(\xi_n)]}
\]

(39)

and

\[
\xi_n = d n + \frac{2x[\cos(d)-1]}{C} + C_2 t + \xi_0
\]

**CONCLUSION**

In this paper, we put a direct method to calculate the rational solitary wave solutions some nonlinear difference differential equations via the relativistic Toda lattice system and the discrete (2+1)-dimensional Toda lattice equations. As a result, many new and more rational solitary wave solutions are obtained, from which hyperbolic function solutions and trigonometric function.

**REFERENCES**


