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Real Quadratic Irrational Numbers under the Action of Hecke Groups

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Abstract: $\mathbb{Q}^*(\sqrt{k^2m})$ is the set of all roots of primitive second degree equations $ct^2+2at+b=0$, with reduced discriminant $\Delta = a^2$-bc equal to k^2m and $\mathbb{Q}(\sqrt{m})\backslash\mathbb{Q}$ is the disjoint union of $\mathbb{Q}^*(\sqrt{k^2m})$, for all $k\in\mathbb{Z}$. The Authors study the action of two hecke groups- the full modular group $H(\lambda_3)=PSL_2(\mathbb{Z})$ and the group of Möbius transformations $H(\sqrt{2})=\langle x,y:x^2=y^4=1\rangle$. The action of $H(\lambda_3)=G$, $H(\sqrt{2})=H$, $G\cap H$ and their subgroups on different sets of numbers $\frac{a+\sqrt{n}}{c}$ with fixed discriminants in the quadratic field has discussed for finding orbits.

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INTRODUCTION

A non-empty set Ω with an action of the group G on it, is said to be a G-set. We say that Ω is a transitive G-set if, for any p,q in Ω there exists a g in G such that $p^g = q$. Since every element of

$$\mathbb{Q}(\sqrt{m})\backslash\mathbb{Q}=\{t+w\sqrt{m}:t,0\neq w\in\mathbb{Z}\}$$

can be expressed uniquely as $\frac{a+\sqrt{n}}{c}$, where $n=k^2m$, k is any positive integer and $a,\frac{a^2-n}{c}$ and c are relatively prime integers and we denote it by $\alpha(a,b,c)$. Then

$$\mathbb{Q}^*(\sqrt{n})=\{\frac{a+\sqrt{n}}{c}:a,c,b=\frac{a^2-n}{c}\in\mathbb{Z}\ |(a,b,c)=1\}$$

is a proper G-subset of $\mathbb{Q}(\sqrt{m})\cup\{\infty\}$ and since $\mathbb{Q}^*(\sqrt{n})\cap\mathbb{Q}^*(\sqrt{n'})=\phi$ for distinct n,n' non-square integers so $\mathbb{Q}(\sqrt{m})\backslash\mathbb{Q}$ is the disjoint union of $\mathbb{Q}^*(\sqrt{k^2m})$ for all $k\in\mathbb{Z}$. If $\alpha(a,b,c)\in\mathbb{Q}^*(\sqrt{n})$ and its conjugate $\bar{\alpha}$ have opposite signs then α is called an ambiguous number [1, 2].

In 1936 Erich Hecke [3] introduced the groups $H(\lambda)$ generated by two linear-fractional transformations

$$x(z)=\frac{-1}{z}\ \text{and}\ y(z)=\frac{-1}{z+\lambda}$$

Hecke showed that $H(\lambda)$ is discrete if and only if $\lambda=\lambda_q=2\cos(\frac{\pi}{q})$, $q\in\mathbb{Z}$, $q\geq3$ or $\lambda\geq2$. Hecke group $H(\lambda_q)$ is isomorphic to the free product of two finite cyclic group of order 2 and q and it has a presentation

$$H(\lambda_q)=\langle x,y:x^2=y^q=1\rangle\cong C_2*C_q$$

The first few of these groups are $H(\lambda_3)=PSL_2(\mathbb{Z})$, the modular group,

$$H(\lambda_4)=H(\sqrt{2})=\langle x,y:x^2=y^4=1\rangle$$

where

$$x(z)=\frac{-1}{2z}\ \text{and}\ y(z)=\frac{-1}{2(z+1)},\ H(\lambda_5)=H(\frac{1+\sqrt{5}}{2})$$

and

$$H(\lambda_6)=H(\sqrt{3})=\langle x,y:x^2=y^6=1\rangle$$

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One of the main reasons for $H(\sqrt{2})$ and $H(\sqrt{3})$ to be two of the most important Hecke groups is that part of the modular group, they are the only Hecke groups $H(\lambda_q)$ whose elements can be completely described. It has been shown [4, 5] that the action of the full modular group $G=\langle x', y' : x'^2 = y'^3 = 1\rangle$, where $x'(z)=\frac{-1}{z}$ and $y'(z)=\frac{-1}{z+1}$, (respectively $H=\left\langle x, y : x^2 = y^4 = 1\right\rangle$, where $x(z)=\frac{-1}{2z}$ and $y(z)=\frac{-1}{2(z+1)}$) on the rational projective line $\mathbb{Q}\cup\{\infty\}$ is transitive. The action of G on $\mathbb{Q}(\sqrt{m})\cup\{\infty\}$ has been discussed in [6, 7]. An action of H and its proper subgroup on $\mathbb{Q}(\sqrt{m})\cup\{\infty\}$ has been discussed in [8-10].

We are concerned with two hecke groups- the full modular group $G=PSL_2(\mathbb{Z})$ and another group of Möbius transformations $H(\sqrt{2})=H$. Different sets of numbers $\frac{a+\sqrt{n}}{c}$ with fixed discriminants in the quadratic field are considered and are looked at different orbits of the action of G,H, $G\cap H$ and their subgroups on these sets. The results of earlier papers on the numbers of orbits and the properties of elements belonging to them are extended by similar results related to the new twist connected to the group H which has nontrivial intersection with G and opens a possibility to look at orbits which were not computed in earlier paper. We consider one of the interesting subgroups of $G\cap H$ is $H^* = \langle xy, yx\rangle$. In Section 3 we prove that for each H^*-subset A of $\mathbb{Q}^{**}(\sqrt{n})$ or $\mathbb{Q}^{*}(\sqrt{n})\setminus\mathbb{Q}^{**}(\sqrt{n})$, $A\cup x'(A)$ is a G-subset of $\mathbb{Q}^{*}(\sqrt{n})$. We also prove that for each H^*-subset A of $\mathbb{Q}^{**}(\sqrt{n})$, $A\cup x(A)$ is an H-subset of $\mathbb{Q}^{**}(\sqrt{n})$ or $\mathbb{Q}^{*\sim}(\sqrt{n})$ according as $n\not\equiv 0 \pmod 4$ or $n\equiv 0 \pmod 4$ similarly if A is an H^*-subset of $\mathbb{Q}^{*}(\sqrt{n})\setminus\mathbb{Q}^{**}(\sqrt{n})$ then $A\cup x(A)$ is an H-subset of $\mathbb{Q}^{*\sim}(\sqrt{4n})$ for each non-square n. In particular, the partition of $\mathbb{Q}^{*}(\sqrt{n})$ has been discussed depending upon classes [a,b,c] modulo p_1p_2.

PRELIMINARIES

We quote from [6-8, 10] the following results for later reference. Also we tabulate the actions on $\alpha(a,b,c)\in\mathbb{Q}^{*}(\sqrt{n})$ of x', y' and x,y, the generators of G and H respectively in Table 1 and 2.

Lemma 2.1: [8] Let $\alpha(a,b,c)\in\mathbb{Q}^{*}(\sqrt{n})$. Then:

Table 1: The action of elements of G on $\alpha\in\mathbb{Q}^{*}(\sqrt{n})$

$x'(\alpha)=\frac{-1}{\alpha}$	-a	c	b
$y'(\alpha)=\frac{\alpha-1}{\alpha}$	-a +b	-2a+b+c	c
$(y')^2(\alpha)=\frac{1}{1-\alpha}$	-a +c	c	-2a+b+c
$x'y'(\alpha)=\frac{\alpha}{1-\alpha}$	a-b	b	-2a+b+c
$y'x'(\alpha)=1+\alpha$	a+c	2a+b+c	c
$(y')^2x'(\alpha)=\frac{\alpha}{1+\alpha}$	a+b	b	2a+b+c

Table 2: The action of elements of H on $\alpha(a,b,c)\in\mathbb{Q}^{*}(\sqrt{n})$

$x(\alpha)=\frac{-1}{2\alpha}$	-a	$\frac{c}{2}$	2b
$y(\alpha)=\frac{-1}{2(\alpha+1)}$	-a-c	$\frac{c}{2}$	2(2a+b+c)
$y^2(\alpha)=\frac{-(\alpha+1)}{2(\alpha)}$	-3a-2b-c	2a+b+c	4a+4b+c
$y^3(\alpha)=\frac{(2\alpha+1)}{2\alpha}$	-a-2b	$\frac{4a+4b+c}{2}$	2(2a+b+c)
$xy(\alpha)=\alpha+1$	a+c	2a+b+c	c
$yx(\alpha)=\frac{\alpha}{1-2\alpha}$	a-2b	b	-4a+4b+c
$y^2x(\alpha)=\frac{1-2\alpha}{2(-1+\alpha)}$	3a-2b-c	$\frac{-4a+4b+c}{2}$	2(-2a+b+c)
$y^3x(\alpha)=\alpha-1$	a-c	2a+b+c	c

1. If $n\not\equiv 0 \pmod 4$ then $\frac{\alpha}{2}\in\mathbb{Q}^{**}(\sqrt{n})$ if and only if $2|b$.
2. $\frac{\alpha}{2}\in\mathbb{Q}^{**}(\sqrt{4n})$ if and only if $2\nmid b$.

Theorem 2.2: [8] The set

$$\mathbb{Q}''(\sqrt{n})=\{\frac{\alpha}{t}:\alpha\in\mathbb{Q}^{*}(\sqrt{n}),\ t=1,2\}$$

is invariant under the action of H.

Theorem 2.3: [8] For each non square positive integer $n\equiv 1,2$ or $3 \pmod 4$, $\mathbb{Q}^{**}\sqrt{n})=\{\alpha\in\mathbb{Q}^{*}(\sqrt{n}):2|c\}$ is an H-subset of $\mathbb{Q}''(\sqrt{n})$.

Theorem 2.4: [6] If $n\equiv 0$ or $3 \pmod 4$, then

$$S=\{\alpha\in\mathbb{Q}^{*}(\sqrt{n}):b \text{ or } c\equiv 1 \pmod 4\}$$

and

$$-S = \{\alpha \in \mathbb{Q}^*(\sqrt{n}) : b \text{ or } c \equiv -1 (\text{mod } 4)\}$$

are exactly two disjoint G-subsets of $\mathbb{Q}^*(\sqrt{n})$ depending upon classes [a,b,c] modulo 4.

Theorem 2.5: [6] If $n \equiv 1(\text{mod } 4)$, then $\mathbb{Q}'(\sqrt{n}) = \{\alpha \in \mathbb{Q}^*(\sqrt{n}) : 2 | (b,c)\}$ and $\mathbb{Q}^*(\sqrt{n}) \setminus \mathbb{Q}'(\sqrt{n})$ are both G-subsets of $\mathbb{Q}^*(\sqrt{n})$.

Theorem 2.6: [10] Let $n \equiv 2(\text{mod } 8)$, $n \neq 2$. Then

$$B^1 = \{\alpha \in \mathbb{Q}^*(\sqrt{n}) : b \text{ or } c \equiv \pm 1 (\text{mod } 8)\}$$

and

$$B^3 = \{\alpha \in \mathbb{Q}^*(\sqrt{n}) : b \text{ or } c \equiv \pm 3 (\text{mod } 8)\}$$

are both G-subsets of $\mathbb{Q}^*(\sqrt{n})$.

Theorem 2.7: [10] Let $n \equiv 6(\text{mod } 8)$. Then

$$B = \{\alpha \in \mathbb{Q}^*(\sqrt{n}) : b \text{ or } c \equiv 1 \text{ or } 3 (\text{mod } 8)\}$$

and

$$-B = \{\alpha \in \mathbb{Q}^*(\sqrt{n}) : b \text{ or } c \equiv -1 \text{ or } -3 (\text{mod } 8)\}$$

are both G-subsets of $\mathbb{Q}^*(\sqrt{n})$.

Theorem 2.8: [7] Let p be an odd prime factor of n. Then both of

$$S_1^p = \{\alpha \in \mathbb{Q}^*(\sqrt{n}) : (b/p) \text{ or } (c/p) = 1\}$$

and

$$S_2^p = \{\alpha \in \mathbb{Q}^*(\sqrt{n}) : (b/p) \text{ or } (c/p) = -1\}$$

are G-subsets of $\mathbb{Q}^*(\sqrt{n})$. In particular, these are the only G-subsets of $\mathbb{Q}^*(\sqrt{n})$ depending upon classes [a,b,c] modulo p.

ACTION OF $PSL_2(\mathbb{Z}) \cap H(\sqrt{2})$ ON $\mathbb{Q}^*(\sqrt{n})$

We start this section by recalling that

$$G = \langle x', y' : x'^2 = y'^3 = 1 \rangle, \quad H = \langle x, y : x^2 = y^4 = 1 \rangle$$

where

$$x'(\alpha) = \frac{-1}{\alpha}, \quad y'(\alpha) = \frac{\alpha - 1}{\alpha}, \quad x(\alpha) = \frac{-1}{2\alpha}$$

and

$$y(\alpha) = \frac{-1}{2(\alpha + 1)}$$

The proper subset $\mathbb{Q}^*(\sqrt{n})$ of $\mathbb{Q}(\sqrt{m}) \setminus \mathbb{Q}$ is invariant under the action of modular group G but $\mathbb{Q}^*(\sqrt{n})$ is not invariant under the mobious group H. Thus it motivates us to establish a connection between the elements of the groups G and H and hence to deduce a common subgroup $H^* = \langle xy, yx \rangle$ of both groups such that each of $\mathbb{Q}^{**}(\sqrt{n})$ and $\mathbb{Q}^*(\sqrt{n}) \setminus \mathbb{Q}^{**}(\sqrt{n})$ is invariant under H^* and hence we find G-subsets of $\mathbb{Q}^*(\sqrt{n})$ and H-subsets of $\mathbb{Q}^{**}(\sqrt{n})$, $\mathbb{Q}^{*\sim}(\sqrt{n})$ according as $n \not\equiv 0(\text{mod } 4)$ or $n \equiv 0$ (mod 4) and $\mathbb{Q}^{*\sim}(\sqrt{4n})$ for all non-square n.

For this we need the following crucial results which show the relationships between the elements of G and H.

Lemma 3.1 Let x', y' and x,y be the generators of G and H respectively defined above. Then we have:

1. $y^2 = (x'y')^2(y'x')$ and $y^3 = \frac{1}{2}(x'(y')^2)^2 x'$.
2. $xy = y'x'$ and $yx = (x'y')^2$.
3. $y^3 x = x'(y')^2$ and $xy^3 = ((y')^2 x')^2$
4. $y^2 x = \frac{1}{2}(x'(y')^2)(x'y')$ and $xy^2 = \frac{1}{2}(y'x')((y')^2 x')$.
5. $x' = 2x$ and $y' = (2x)(2y)(2x)$.
6. $x'y' = 2(yx)\frac{1}{2}$ and $x'(y')^2 = y^3 x$. In particular $(x'y')^{-1} = 2(xy^3)\frac{1}{2}$ and $(x'(y')^2) = xy$.

Following corollaries are immediate consequences of Lemma 3.1.

Corollaries 3.2

1. By Lemma 3.1, since $xy = y'x'$ and $yx = (x'y')^2$ so $H^* = \langle xy, yx \rangle$ is a common subgroup of G and H where xy, yx are the transformations defined by $xy(\alpha) = \alpha + 1$ and $yx(\alpha) = \frac{\alpha}{1 - 2\alpha}$.
2. As $yxxy = y^2$, $xyyx = xy^2x$, so $\langle y^2, xy^2x \rangle$ is a proper subgroup of H^*.
3. $\langle H^*, x \rangle = \langle H^*, y \rangle = H$ and $\langle H^*, x' \rangle = \langle H^*, y' \rangle = G$.

Since for each integer n, either (n/p) = 0 or (n/p) = ±1 for each odd prime p. So in the following lemma we classify the elements of $\mathbb{Q}^*(\sqrt{n})$ in terms of classes [a,b,c](mod p) with 0 modulo p or qr, qnr nature of a,b and c modulo p.

Lemma 3.3: Let p be prime and n ≡ 0 (mod p). Then E_p^0 consists of classes [0, 0, qr], [0, 0, qnr], [0, qr, 0], [0, qnr, 0], [qr, qr, qr], [qnr, qr, qr], [qr, qnr, qnr] or [qnr, qnr, qnr].

Proof: Let [a,b,c](mod p) be any class of $\alpha_n(a,b,c)$. Then $a^2 \equiv bc$ (mod p) leads us to exactly three cases. If a ≡ 0 (mod p) then exactly one of b,c is ≡ 0 (mod p) and the other is qr or qnr of p as otherwise (a,b,c)≠1 and hence the class [a,b,c] is one of the forms [0, 0, qr], [0, 0, qnr], [0, qr, 0], [0, qnr, 0]. If (a/p) = 1 then (bc/p) = 1 and the class takes the form [qr, qr, qr] or [qr, qnr, qnr]. In third case if (a/p) = -1 then $(a^2/p) = 1$ so again (bc/p) = 1. This yields the class in the forms [qnr, qr, qr] or [qnr, qnr, qnr]. Hence the result.

Lemma 3.4: Let (n/p) = 1 and let [a,b,c](mod p) be the class of $\alpha_n(a,b,c)$ of $\square^*(\sqrt{n})$. Then:

1. If p ≡ 1 (mod 4) then [a,b,c](mod p) has the forms [0, qr, qr], [0, qnr, qnr], [qr, 0, qr], [qr, 0, qnr], [qr, qr, 0],, [qr, qnr, 0], [qnr, 0, qr], [qnr, 0, qnr], [qnr, qr, 0], [qnr, qnr, 0], [qnr, 0, 0] or [qr, 0, 0] only.
2. If p ≡ 3 (mod 4) then [a,b,c](mod p) has the forms [0, qnr, qr], [0, qr, qnr], [qr, 0, qr], [qr, qr, 0], [qr, 0, qnr], [qr, qnr, 0], [qnr, 0, qr], [qnr, qr, 0], [qnr, 0, qnr], [qnr, qnr, 0], [qnr, 0, 0] or [qr, 0, 0] only.

Proof: Let [a,b,c](mod p) be the class of $\alpha_n(a,b,c)$ with a^2-n = bc. As (n/p) = 1 so if (a/p) = 0 then $((a^2-n)/p)$ = ±1 according as p ≡ 1 (mod 4) or p ≡ 3 (mod 4). Thus we have [0, qr, qr], [0, qnr, qnr] if p ≡ 1 (mod 4) and [0, qnr, qr], [0, qr, qnr] if p ≡ 3 (mod 4). If (a/p) = ±1 then $((a^2-n)/p) = 0$, so we get [qr, 0, qr], [qr, 0, qnr], [qr, qr, 0],, [qr, qnr, 0], [qnr, 0, qr], [qnr, 0, qnr], [qnr, qr, 0], [qnr, qnr, 0], [qnr, 0, 0] or [qr, 0, 0] only. This completes the proof.

Lemma 3.5 Let (n/p) = -1 and let [a,b,c](mod p) be the class of $\alpha(a,b,c)$ of $\square^*(\sqrt{n})$. Then:

1. If p ≡ 1 (mod 4) then [a,b,c](mod p) has the forms [0, qnr, qr], [0, qr, qnr], [qr, qr, qr], [qr, qnr, qnr], [qnr, qr, qr] or [qnr, qnr, qnr] only.
2. If p ≡ 3 (mod 4) then [a,b,c](mod p) has the forms [0, qr, qr], [0, qnr, qnr], [qr, qr, qnr], [qr, qnr, qr], [qnr, qr, qnr] or [qnr, qnr, qr] only.

Proof: Let [a,b,c](mod p) be any class of $\alpha_n(a,b,c)$ with a^2-n = bc. As (n/p) = -1 so if (a/p) = 0 then by $((a^2-n)/p)$ = -1 or $((a^2-n)/p) = 1$ according as p ≡ 1 (mod 4) or p ≡ 3 (mod 4). Thus we have [0, qnr, qr], [0, qr, qnr] or [0,qr,qr], [0, qnr, qnr]. If (a/p) = ±1 then $((a^2-n)/p) = \pm 1$. Therefore p ≡ 1 (mod 4) gives [qr, qr, qr], [qr, qnr, qnr], [qnr, qr, qr] or [qnr, qnr, qnr] and if p ≡ 3 (mod 4), then we have [qr, qnr, qr], [qr, qr, qnr], [qnr, qnr, qr] or [qnr, qr, qnr].

Note: [6] If (n/2) = 0 then [1,1,1], [0,0,1] and [0,1,0] are three classes of $\square^*(\sqrt{n})$ in modulo 2. If n is an odd then three classes of $\square^*(\sqrt{n})$ are [1,0,1], [1,1,0] and [0,1,1] modulo 2. These are the only classes of $\square^*(\sqrt{n})$ if n ≡ 3 (mod 4). But if n ≡ 1 (mod 4) then [1,0,0] is also a class of $\square^*(\sqrt{n})$ and there are no further classes. These classes in modulo 2 of $\square^*(\sqrt{n})$ do not give any useful information during the study of action of G on $\square^*(\sqrt{n})$ except that if n ≡ 1(mod 4) then the set consisting of all elements of $\square^*(\sqrt{n})$ of the form [1,0,0] is invariant under the action of the group G. Whereas the study of action of H^* on $\square^*(\sqrt{n})$ gives some useful information about these classes. The following crucial result determines the H^*-subsets of $\square^*(\sqrt{n})$ depending upon classes [a,b,c] modulo 2. It is interesting to observe that $\square^*(\sqrt{n})$ splits into $\square^{**}(\sqrt{n})$ and $\square^*(\sqrt{n})\setminus\square^{**}(\sqrt{n})$ in modulo 2. Each of these two H^*-subsets further splits into proper H^*-subsets in modulo 4.

Lemma 3.6: $\square^{**}(\sqrt{n})$ and $\square^*(\sqrt{n})\setminus\square^{**}(\sqrt{n})$ are two distinct H^*-subsets of $\square^*(\sqrt{n})$ depending upon classes [a,b,c] modulo 2.

Theorem 3.7 and Remarks 3.8 are extension of Lemma 3.6 and discuss the action of H^* on $\square^*(\sqrt{n})$ depending upon classes [a,b,c] modulo 4. Proofs of these follow directly by the equations

$$xy\left(\frac{a+\sqrt{n}}{c}\right) = \frac{a+c+\sqrt{n}}{c}, \quad yx\left(\frac{a+\sqrt{n}}{c}\right) = \frac{a-2b+\sqrt{n}}{-4a+4b+c}$$

and classes [a,b,c] modulo 4 given in [6].

Theorem 3.7: Let n be any non-square positive integer. Then $\square^*(\sqrt{n})\setminus\square^{**}(\sqrt{n})$ splits into two proper H^*-subsets

$$A_1 = \{\alpha \in \square^*(\sqrt{n})\setminus\square^{**}(\sqrt{n}) : c \equiv 1 (\text{mod } 4)\}$$

$$A_2 = \{\alpha \in \square^*(\sqrt{n})\setminus\square^{**}(\sqrt{n}) : c \equiv 3 (\text{mod } 4)\}$$

Similarly $\square^{**}(\sqrt{n})$ splits into two proper H^*-subsets

$$B_1 = \{\alpha \in \square^{**}(\sqrt{n}) : c \equiv 0 (\text{mod } 4)\}$$

and

$$B_2 = \{\alpha \in \square^{**}(\sqrt{n}) : c \equiv 2 (\text{mod } 4)\}$$

Remarks 3.8

1. Let $n \equiv 1$ (mod 4). Then

$$\square'(\sqrt{n}) = \{\alpha \in \square^{**}(\sqrt{n}) : 2 | (b,c)\}$$

and $\square^{**}(\sqrt{n}) \setminus \square'(\sqrt{n})$ are H^*-subsets of $\square^{**}(\sqrt{n})$. In particular if $n \equiv 5$ (mod 8), then $B_1 = \square^{**}(\sqrt{n}) \setminus \square'(\sqrt{n})$ and $B_2 = \square'(\sqrt{n})$ are H^*-subsets of $\square^{**}(\sqrt{n})$. Whereas if $n \equiv 1$ (mod 8), then

$$C_1 = \{\alpha \in \square'(\sqrt{n}) \cap B_1 : a \equiv 1 (\text{mod } 4)\}$$

$$C_2 = \{\alpha \in \square'(\sqrt{n}) \cap B_1 : a \equiv 3 (\text{mod } 4)\}$$

$$C_3 = \{\alpha \in \square'(\sqrt{n}) : c \equiv 2 (\text{mod } 4)\}$$

and

$$C_4 = \square^{**}(\sqrt{n}) \setminus \square'(\sqrt{n})$$

are H^*-subsets of $\square^{**}(\sqrt{n})$. Specifically, $B_1 = C_1 \cup C_2 \cup C_4$, $B_2 = C_3$.

3. As we know that if n and c are even, then a must be even as (a,b,c) = 1. If $n \equiv 2$ (mod 4), then $B_2 = \square^{**}(\sqrt{n})$ and $B_1 = \phi$.
4. If $n \equiv 0$ or 3 (mod 4), then B_2 or B_1 is empty according as $n \equiv 0$ or 3 (mod 4). As we know that if n and c are even, then a must be even as (a,b,c) = 1. However $D_1 = \{\alpha \in \square^{**}(\sqrt{n}) : b \equiv 1 (\text{mod } 4)\}$, $D_2 = \{\alpha \in \square^{**}(\sqrt{n}) : b \equiv 3 (\text{mod } 4)\}$ are proper H^*-subsets of $\square^{**}(\sqrt{n})$ depending upon classes [a,b,c] modulo 4.

Lemma 3.9: Let n be any non-square positive integer. Then $\square^{**}(\sqrt{4n})$ and $\square^{*}(\sqrt{n}) \setminus \square^{**}(\sqrt{n})$ are distinct H^*-subsets of an H-set

$$\square^{*\sim}(\sqrt{4n}) = \square^{**}(\sqrt{4n}) \cup \left(\square^{*}(\sqrt{n}) \setminus \square^{**}(\sqrt{n})\right)$$

Proof: Follows by the equations $x\left(\square^{*}(\sqrt{n}) \setminus \square^{**}(\sqrt{n})\right) = \square^{**}(\sqrt{4n})$ and vice versa. Hence $\left(\square^{*}(\sqrt{n}) \setminus \square^{**}(\sqrt{n})\right)$ is equivalent to $\square^{**}(\sqrt{4n})$.

Clearly $\left|\square_1^{*}(\sqrt{n}) \setminus \square_1^{**}(\sqrt{n})\right| = \left|\square_1^{**}(\sqrt{4n})\right|$ where $\square_1^{*}(\sqrt{n})$ denotes the set of all ambigious numbers in $\square^{*}(\sqrt{n})$ [2].

Remark 3.10

(i) Each G-subset X of $\square^{*}(\sqrt{n})$ splits into two H^*-subsets $X \setminus \square^{**}(\sqrt{n})$ and $X \cap \square^{**}(\sqrt{n})$ and

$$x'\left(X \setminus \square^{**}(\sqrt{n})\right) = x'\left(X \cap \square^{**}(\sqrt{n})\right) = X$$

(ii) Each H-subset Y of $\square^{*\sim}(\sqrt{4n})$ splits into two H^*-subsets $Y \setminus \square^{**}(\sqrt{n})$ and $Y \cap \square^{**}(\sqrt{4n})$.

(iii) Each H-subset Y of $\square^{*\sim}(\sqrt{n})$, $n \not\equiv 0 (\text{mod } 4)$ splits into two H^*-subsets $Y \setminus \square^{**}(\sqrt{n})$ and $Y \cap \square^{**}(\sqrt{4n})$.

(iv) Each H-subset Y of $\square^{**}(\sqrt{n})$, $n \not\equiv 0 (\text{mod } 4)$ splits into two H^*-subsets $Y \setminus \square^{**}(\sqrt{n})$ and $Y \cap \square^{**}(\sqrt{4n})$.

Theorem 3.11

(a) If A is an H^*-subset of $\square^{**}(\sqrt{n})$ or $\left(\square^{*}(\sqrt{n}) \setminus \square^{**}(\sqrt{n})\right)$, then $A \cup x'(A)$ is a G-subset of $\square^{*}(\sqrt{n})$.

(b) If A is an H^*-subset of $\square^{**}(\sqrt{n})$, then $A \cup x(A)$ is an H-subset of $\square^{**}(\sqrt{n})$ or $\square^{*\sim}(\sqrt{n})$) according as $n \not\equiv 0 (\text{mod } 4)$ or $n \equiv 0$ (mod 4)).

(c) If A is an H^*-subset of $\left(\square^{*}(\sqrt{n}) \setminus \square^{**}(\sqrt{n})\right)$, then $A \cup x(A)$ is an H-subset of $\square^{*\sim}(\sqrt{4n})$ for all non-square n.

Proof: Proof of (a) follows by the equation

$$x'(\square^{**}(\sqrt{n})) = \square^{*}(\sqrt{n}) \setminus \square^{**}(\sqrt{n})$$

Proof of (b) follows by the equations

$$x(\square^{**}(\sqrt{n})) = \square^{**}(\sqrt{n}) \text{ or } x(\square^{**}(\sqrt{n})) = \square^{*}\left(\sqrt{\frac{n}{4}}\right) \setminus \square^{**}\left(\sqrt{\frac{n}{4}}\right)$$

according as $n \not\equiv 0 (\text{mod } 4)$ or $n \equiv 0$ (mod 4)).

Proof of (c) follows by the equation

$$x\left(\square^{*}(\sqrt{n}) \setminus \square^{**}(\sqrt{n})\right) = \square^{**}(\sqrt{4n})$$

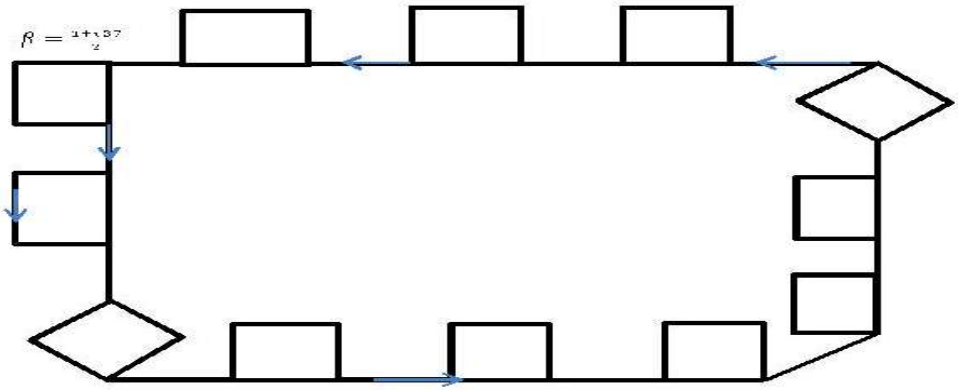

Fig. 1: Orbit of $\beta = \frac{1+\sqrt{37}}{2}$ and $(xy)^3(xy^2)(xy^3)^5(xy^2)(xy)^2(\beta) = \beta$

Following examples illustrate the above results.

Examples 3.12

1. Let n = 8. Then

$$\alpha = \frac{1+\sqrt{8}}{1} \in \square^{*}(\sqrt{8})$$

but

$$\frac{\alpha}{2} = \frac{1+\sqrt{8}}{2} = \frac{2+\sqrt{32}}{4} \in \square^{**}(\sqrt{32})$$

Also

$$\beta = \frac{2+\sqrt{8}}{1} \in \square^{*}(\sqrt{8})$$

but

$$\frac{\beta}{2} = \frac{1+\sqrt{2}}{1} \in \square^{*}(\sqrt{2}) \setminus \square^{**}(\sqrt{2})$$

Similarly

$$\gamma = \frac{2+\sqrt{8}}{4} \in \square^{**}(\sqrt{8})$$

whereas

$$\frac{\gamma}{2} = \frac{4+\sqrt{32}}{16} \in \square^{*}(\sqrt{32})$$

Also

$$\square^{*\sim}(\sqrt{8}) = (\sqrt{2})^H \cup (-\sqrt{2})^H$$

$$\square^{*\sim}(\sqrt{32}) = (\sqrt{8})^H \cup (-\sqrt{8})^H$$

So $\square''(\sqrt{8})$ has exactly 4 orbits under the action of H whereas $\square^{*}(\sqrt{8})$ splits into two G-orbits namely $(\sqrt{8})^G$, $(-\sqrt{8})^G$.

2. $\square''(\sqrt{37})$ splits into nine H-orbits. Also

$$\square^{*\sim}(\sqrt{148}) = (\sqrt{37})^H \cup (-\sqrt{37})^H \cup (\frac{1+\sqrt{37}}{3})^H \cup (\frac{1+\sqrt{37}}{-3})^H \cup (\frac{-1+\sqrt{37}}{-3})^H \cup (\frac{-1+\sqrt{37}}{-3})^H$$

and

$$\square^{**}(\sqrt{37}) = (\frac{1+\sqrt{37}}{2})^H \cup (\frac{1+\sqrt{37}}{4})^H \cup (\frac{-1+\sqrt{37}}{-4})^H$$

whereas $\square^{*}(\sqrt{37})$ splits into four G-orbits namely $(\sqrt{37})^G$

$$(\frac{1+\sqrt{37}}{2})^G \quad (\frac{1+\sqrt{37}}{3})^G \text{ and } (\frac{-1+\sqrt{37}}{-3})^G$$

In [7] it was proved that S_1^p, S_2^p are G-subsets of $\square^{*}(\sqrt{n})$ for each odd prime factor of n. We now study the action of H^* on $\square^{*}(\sqrt{n})$.

Theorem 3.13: Let p be an odd prime factor of n. Then

$$S_1^p = \{\alpha \in \square^{*}(\sqrt{n}) : (b/p) \text{ or } (c/p) = 1\}$$

and

$$S_2^p = \{\alpha \in \square^{*}(\sqrt{n}) : (b/p) \text{ or } (c/p) = -1\}$$

are two H^*-subsets of $\square^{*}(\sqrt{n})$. In particular, these are the only H^*-subsets of $\square^{*}(\sqrt{n})$ depending upon classes [a,b,c] modulo p.

Proof: Let [a,b,c](mod p) be the class of $\alpha(a,b,c) \in \square^{*}(\sqrt{n})$. In view of Lemma 3.3, either both of b,c are qrs or qnrs and the two equations

$$xy(\alpha(a,b,c)) = \alpha'(a+c,\ 2a+b+c,\ c)$$

$$yx(\alpha(a,b,c)) = \alpha'(a-2b,\ b,\ -4a+4b+c)$$

fix b,c modulo p. If a ≡ b ≡ 0(mod p) then ((2a+b+c)/p) = 1 or ((2a+b+c)/p) = -1 according as (c/p) = 1 or

(c/p) = -1. similarly for $a \equiv c \equiv 0 \pmod p$. This shows that the sets S_1^p and S_2^p are H^*-subsets of $\square^*(\sqrt{n})$ depending upon classes modulo p.

The following corollary is an immediate consequence of Lemma 3.6 and Theorem 3.13.

Corollary 3.14: Let p be an odd prime such that $n \equiv 0 \pmod{2p}$. Then $\square^*(\sqrt{n})$ splits into four proper H^*-subsets depending upon classes modulo 2p.

Proof: Since a^2-n = bc implies that $a^2 \equiv bc \pmod p$. This is equivalent to congruences $a^2 \equiv bc \pmod p$ and $a^2 \equiv bc \pmod p$. By Theorem 3.13 S_1^p, S_2^p are H^*-subsets and then, by Lemma 3.6, each of S_1^p and S_2^p further splits into two H^*-subsets $S_1^p \cap \square^{**}(\sqrt{n})$, $S_2^p \cap \square^{**}(\sqrt{n})$, $S_1^p \setminus \square^{**}(\sqrt{n})$ and $S_2^p \setminus \square^{**}(\sqrt{n})$.

The next theorem is more interesting in a sense that whenever (n/p) = ±1, $\square^*(\sqrt{n})$ is itself an H^*-set depending upon classes [a,b,c] modulo p.

Theorem 3.15: Let p be an odd prime and (n/p) = ±1. Then $\square^*(\sqrt{n})$ is itself an H^*-set depending upon classes [a,b,c] modulo p.

Proof: follows from Lemmas 3.4, 3.5 and the equations $xy(\alpha) = \alpha+1$ and $yx(\alpha) = \dfrac{\alpha}{1-2\alpha}$ given in Table 2.

Let us illustrate the above theorem in view of Theorem 3.4. If (n/3) = 1, then the set {[0,1,2], [0,2,1], [1,0,1], [1,1,0], [2,0,2], [2,0,1], [2,1,0], [2,2,0], [1,2,0], [1,0,2], [1,0,0], [2,0,0]} is an H^*-set. That is, $\square^*(\sqrt{n})$ is itself an H^*-set depending upon classes [a,b,c] modulo 3. Similarly for (n/3) = -1.

Theorem 3.16: Let p be an odd prime and n is a quadratic residue (quadratic non-residue) of 2p. Then $\square^*(\sqrt{n})$ is the disjoint union of three H^*-subsets $\left(\square^*(\sqrt{n}) \setminus \square^{**}(\sqrt{n})\right)$, $\square^{**}(\sqrt{n}) \setminus \square'(\sqrt{n})$ and $\square'(\sqrt{n})$ depending upon classes [a,b,c] modulo 2p.

Proof: Follows by Theorems 2.2, 2.3 and 3.15.

The following example justifies the above result.

Example 3.17: Since 17 = 5 (mod 6), then $\square^*(\sqrt{15})$ splits into these three H^*-subsets {[0,1,1], [1,2,1], [2,5,1], [3,4,1], [4,5,1], [5,2,1], [0,5,5], [5,4,5], [4,1,5], [3,2,5], [2,1,5], [1,4,5]} {[1,1,2], [3,5,2], [5,1,2], [3,1,4], [1,5,4], [5,5,4], [1,2,4], [5,2,4], [3,4,4], [1,4,2], [3,2,2], [5,4,2]}

The next theorem is a generalization of Theorem 3.13 to the case when n involves two distinct prime factors.

Theorem 3.20: Let p_1 and p_2 be distinct odd primes factors of n. Then $S_{1,1} = S_1^{p_1} \cap S_1^{p_2}$, $S_{1,2} = S_1^{p_1} \cap S_2^{p_2}$, $S_{2,1} = S_2^{p_1} \cap S_1^{p_2}$ and $S_{2,2} = S_2^{p_1} \cap S_2^{p_2}$ are four H^*-subsets of $\square^*(\sqrt{n})$. More precisely these are the only H^*-subsets of $\square^*(\sqrt{n})$ depending upon classes [a,b,c] modulo p_1p_2.

Proof: Let $[a,b,c] \pmod{p_1p_2}$ be any class of $\alpha(a,b,c) \in \square^*(\sqrt{n})$ with $n \equiv 0 \pmod{p_1p_2}$. Then a^2-n = bc implies that

$$a^2 \equiv bc \pmod{p_1p_2} \qquad (1)$$

This is equivalent to congruences $a^2 \equiv bc \pmod{p_1}$ and $a^2 \equiv bc \pmod{p_2}$. By Theorem 3.14, the congruence $a^2 \equiv bc \pmod{p_1}$ gives two H^*-subsets

$$S_1^{p_1} = \{\alpha \in \square^*(\sqrt{n}) : (c/p_1) \text{ or } (c/p_1) = 1\}$$

and

$$S_2^{p_1} = \{\alpha \in \square^*(\sqrt{n}) : (c/p_1) \text{ or } (c/p_1) = -1\}$$

of $\square^*(\sqrt{n})$. As $a^2 \equiv bc \pmod{p_2}$, again applying Theorem 3.13 on each of $S_1^{p_1}$ and $S_2^{p_1}$ we have four H^*-subsets $S_{1,1}, S_{1,2}, S_{2,1}$ and $S_{2,2}$ of $\square^*(\sqrt{n})$.

CONCLUSION

We studiedthe action of $H^* = \langle xy, yx \rangle$ on different sets of numbers $\dfrac{a+\sqrt{n}}{c}$ with fixed discriminants in the quadratic field and proved that for each H^*-subset A of $\square^{**}(\sqrt{n})$ or $\left(\square^*(\sqrt{n}) \setminus \square^{**}(\sqrt{n})\right)$, $A \cup x'(A)$ is a G-subset of $\square^*(\sqrt{n})$. We also verified that for each H^*-subset A of $\square^{**}(\sqrt{n})$, $A \cup x(A)$ is an H-subset of $\square^{**}(\sqrt{n})$ or $\square^{*\sim}(\sqrt{n})$ according as $n \not\equiv 0 \pmod 4$ or $n \equiv 0 \pmod 4$. Similarly if A is an H^*-subset of $\left(\square^*(\sqrt{n}) \setminus \square^{**}(\sqrt{n})\right)$ then $A \cup x(A)$ will be an H-subset of $\square^{*\sim}(\sqrt{4n})$ for each

non-square n. In particular, if p_1 and p_2 are distinct odd primes factors of n, then we found four H^*-subset of $\mathbb{Q}^*(\sqrt{n})$ depending upon classes [a,b,c] modulo p_1p_2.

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