Common Fixed Point of Fuzzy Mappings on Closed Balls

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Abstract: In this paper, we establish some common fixed point theorems for fuzzy mappings satisfying fuzzy contractive conditions which are general than Chatterjee type and Kannan type fuzzy contractive conditions on closed balls in a complete metric space. Our results generalize the corresponding results in the current literature.

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INTRODUCTION

The notion of fixed point for fuzzy mappings was introduced by Weiss [1] and Butnariu [2]. Heilpern [3] introduced the concept of fuzzy contraction mappings and studied fixed point theorems for fuzzy set-valued mappings. He established Banach contraction principal for fuzzy mappings in complete metric linear spaces. This constitutes a fuzzy extension of Banach fixed point theorem and Nadler’s theorem [4] for multi-valued mappings. Further, many other authors like [5-13] have studied the existence of fixed points and common fixed points of fuzzy mappings satisfying certain contractive type conditions. Frigon and O’Regan [14] proved some fuzzy fixed point theorems on closed balls. In this paper, we establish some fixed point theorems for a pair of fuzzy mappings satisfying contractive conditions more general than Chatterjee type [15] and Kannan type fuzzy mappings [16].

PRELIMINARIES

Let us gather some preliminaries. Let X be a non-empty set. Then

\[ 2^X = \{ A : A \subseteq X \} \]

\[ \text{CL}(2^X) = \{ A \in 2^X : A \text{ is non-empty and closed} \} \]

\[ \text{C}(2^X) = \{ A \in 2^X : A \text{ is non-empty and compact} \} \]

\[ \text{CB}(2^X) = \{ A \in 2^X : A \text{ is non-empty, closed and bounded} \} \]

Let \( d(x,A) = \inf_{y \in A} d(x,y) \) and \( d(A,B) = \inf_{x \in A, y \in B} d(x,y) \).

The Hausdorff metric is defined as

\[ d_H(A,B) = \max \{ \sup_{a \in A} \inf_{b \in B} d(a,b), \sup_{b \in B} \inf_{a \in A} d(a,b) \} \]

A fuzzy set in X is a function with domain S and range [0,1]. We denote by \( I^X \) the collection of all fuzzy sets in X. If A is a fuzzy set and \( x \in X \), then the value \( A(x) \) is called the grade of membership of x in A. The \( \alpha \)-level set of a fuzzy set A is denoted by \( [A]_\alpha \) and is defined by

\[ [A]_\alpha = \{ x : A(x) \geq \alpha \text{ if } \alpha \in (0,1] \} \]

and

\[ [A]_0 = \{ x : A(x) > 0 \} \]

For \( \in X \), we denote the fuzzy set \( \chi_A \) by \( \{ x \} \), where \( \chi_A(x) \) is the characteristic function of the crisp set A.

We now define a subcollection \( F(x) \) of \( I^X \) by

\[ F(x) = \{ A \in I^X : A \text{ is non-empty and closed} \} \]

For \( A, B \in I^X \), \( A \subseteq B \) means \( A(x) \leq B(x) \) for each \( x \in X \) and for \( A, B \in F(X) \), we define

\[ D_I(A,B) = d_H([A],[B]) \]
As in [16], a point \( x^* \in X \) is called a fixed point of fuzzy mapping \( T : X \to \mathbb{I}^X \) if \( \{ x^* \} \subseteq \text{Tx}^* \).

**Lemma 2.1:** [4] Let \( A \) and \( B \) be non-empty closed and bounded subsets of a metric space \((X,d)\). If \( a \in A \), then
\[
d(a,B) \leq d_{H}(A,B)
\]

**Lemma 2.2:** [4] Let \( A \) and \( B \) be non-empty closed and bounded subsets of a metric space \((X,d)\) and \( 0 < \xi < R \). Then for \( a \in A \) there exists \( b \in B \) such that
\[
Hd(a,b) \leq d_{H}(A,B) + \xi
\]

**Remark 2.3:** The completeness of \((X,d)\) implies that \((CB(2^X),d_{H})\) is complete see [17].

**Theorem 2.4:** [16] Let \((X,d)\) be a complete metric space and \( T \) a self map on \( X \). Suppose that there exists a constant \( a \in (0,1/2] \) such that
\[
d(Tx,Ty) \leq (1 + \lambda) d(x,Tx) + (1 + \lambda) d(y,Ty)
\]
holds. Then \( T \) has a unique fixed point in \( X \).

The mappings satisfying contractive condition given in the above theorem are called Kannan mappings. This condition is different from Banach contractive condition \( d(Tx,Ty) \leq \lambda d(x,y) \). Note that the contractive condition in Theorem 2.4 does not make \( T \) continuous whereas Banach contractive condition does.

**MAIN RESULTS**

Let \( \lambda \in X \) and \( 0 < r \in \mathbb{R} \). A ball of radius \( r \) with center at \( x_0 \) is defined as
\[
B_R(x_0) = \{ x \in X : d(x,x_0) < r \}
\]
The closure of \( B_R(x_0) \) is denoted by \( \overline{B_R(x_0)} \).

Now we are all set to prove our main theorem as follows. This establishes the existence of common fixed point for fuzzy mappings satisfying the contractive conditions more general than Chatterjea type [15] and Kannan type [16] contractive condition on closed balls.

**Theorem 3.1:** Let \((X,d)\) be a complete metric space and \( x_0 \in X \). Take two mappings \( F,T : B_R(x_0) \to F(X) \). Suppose that there exists a constant \( k \in (0,1/2) \) with
\[
D_{H}(Fx,Ty) \leq k \max\left(\frac{d(x,[Fx]) + d(y,[Ty])}{d(x,y)}, \frac{d(x,[Fx]) + d(x,y)}{d(x,y)}, \frac{d(y,[Ty]) + d(x,y)}{d(x,y)}\right)
\]
for all \( x,y \in \overline{B_R(x_0)} \) and
\[
d(x_0,[Fx_0]) < (1 - \lambda)r
\]
holds with
\[
\lambda = \max(2k,\frac{k}{1-k})
\]

Then \( F \) and \( T \) has a common fuzzy fixed point in \( B_{R}(x_0) \). That is, there exists \( x^* \in \overline{B_R(x_0)} \) with \( \{ x^* \} \subseteq \text{Fx} \cap \text{Tx} \).

**Proof:** Choose \( \xi \in X \) such that \( \{ x^* \} \subseteq Fx_0 \). This gives \( d(x_0,x^*) < (1 - \lambda)r \) because \( [Fx_0] \neq \emptyset \) and \( d(x_0,[Fx_0]) < (1 - \lambda)r \). This implies that \( x^* \in B_{R}(x_0) \).

Now choose \( \varepsilon > 0 \) such that
\[
\lambda d(x_0,x^*) + \frac{\varepsilon}{1-k} < \lambda(1 - \lambda)r \tag{2}
\]

Then choose \( x_2 \in X \) such that \( \{ x^* \} \subseteq Fx_0 \). By Lemma 2.2,
\[
d(x_1,x_2) \leq D(Fx_0,Tx_1) + \varepsilon \\
\leq k \max\left(\frac{d(x_0,[Fx_0]) + d(x_1,[Tx_1])}{d(x_0,x_1)}, \frac{d(x_0,[Fx_0]) + d(x_1,[Tx_1]) + d(x_0,x_1) + \varepsilon}{d(x_0,x_1)}\right) \\
\leq \max\left(\frac{d(x_0,x_1) + d(x_1,x_2)}{d(x_0,x_1)} \right) + d(x_0,x_1) + d(x_1,x_2) + d(x_0,x_1) + \varepsilon
\]

**Case I:**
\[
d(x_1,x_2) \leq k d(x_0,x_1) + d(x_1,x_2) + \varepsilon \\
\leq \frac{k}{1-k} d(x_0,x_1) + \frac{\varepsilon}{1-k} \leq \lambda d(x_0,x_1) + \frac{\varepsilon}{1-k}
\]

Using (2), we get
\[
d(x_1,x_2) \leq \lambda d(x_0,x_1) + \frac{\varepsilon}{1-k}
\]

**Case II:** When
\[
d(x_1,x_2) \leq 2d(x_0,x_1) + \varepsilon \leq \lambda d(x_0,x_1) + \frac{\varepsilon}{1-k} \leq \lambda d(x_0,x_1) + \frac{\varepsilon}{1-k} \leq \lambda(1 - \lambda)r
\]
by (2).

Thus in any case
\[
d(x_1,x_2) \leq \lambda(1 - \lambda)r
\]
Since
\[ d(x_0, x_2) \leq d(x_0, x_1) + d(x_1, x_2) \leq (1-\lambda)r + \lambda(1-\lambda)r = (1-\lambda)(1+\lambda)r = (1-\lambda)(1+\lambda^2 + \ldots) = r \]

therefore \( x_2 \in \overline{B}_r(x_0) \).

Continuing this process and having chosen \( x_n \) in \( X \) such that

\[ \{x_{2n+1}\} \subseteq Fx_{2n} \quad \{x_{2n+2}\} \subseteq Tx_{2n+1} \]

with

\[ d(x_{2n+1}, x_{2n+2}) < \lambda^{2n+1}(1-\lambda)r, n=0,1,2,\ldots \]

It is not difficult to see that \( \{x_n\} \) is a Cauchy sequence in \( \overline{B}_r(x) \). As \( \overline{B}_r(x) \) is complete, there exist \( x^* \in \overline{B}_r(x) \) such that \( \lim_{n \to \infty} x_n = x^* \). We are only left with \( \{x^*\} \subseteq Tx^* \) and \( \{x^*\} \subseteq Fx^* \) to prove. Now by Lemma 2.1, we get

\[ d(x^*, [Tx^*]) \leq d(x^*, x_{2n+1}) + d(x_{2n+1}, [Tx^*]) \leq d(x^*, x_{2n+1}) + D(Fx_{2n}, Tx^*) \]

Using (1), we get

\[ d(x^*, [Tx^*]) \leq d(x^*, x_{2n+1}) + \max \left( d(x_{2n}, [Fx_{2n}]), d(x_{2n}, [Tx^*]), d(x_{2n}, x^*), d(x^*, [Tx^*]), d(x_{2n+1}, [Tx^*]) \right) \]

\[ \leq d(x^*, x_{2n+1}) + \max \left( d(x_{2n}, x_{2n+1}), d(x_{2n}, x^*), d(x_{2n+1}, x^*), d(x^*, [Tx^*]), d(x_{2n+1}, [Tx^*]) \right) \]

Taking limit as \( n \to \infty \), we have

\[ d(x^*, [Tx^*]) \leq d(x^*, x^*) + \max \left( d(x^*, x^*), d(x^*, [Tx^*]), d(x^*, x^*), d(x^*, x^*), d(x^*, [Tx^*]), d(x^*, x^*) \right) \leq kd(x^*, [Tx^*]) \]

This gives

\[ (1-k)d(x^*, [Tx^*]) \leq 0 \]

so that \( d(x^*, [Tx^*]) = 0 \) and hence \( \{x^*\} \subseteq Tx^* \).

Similarly, by considering

\[ d(x^*, [Fx^*]) \leq d(x^*, x_{2n+2}) + d(x_{2n+2}, [Fx^*]) \]

We can show that \( \{x^*\} \subseteq Fx^* \). Consequently, the mappings \( F \) and \( T \) have a common fixed point \( x^* \) in \( \overline{B}_r(x) \). That is, \( \{x^*\} \subseteq Fx^* \cap Tx^* \).

**Corollary 3.2** Let \( (X,d) \) be a complete metric space and \( x_0 \in X \). Take two mappings \( F, T: \overline{B}_r(x_0) \to F(X) \).

Suppose there exists a constant \( k \in (0,1/2) \) with

\[ D_l(Fx, Ty) \leq k(d(x, [Fx]) + d(y, [Ty])) \]

for all \( x,y \in \overline{B}_r(x_0) \) and

\[ d(x_0, [Fx_0]) < (1-\lambda)r \]

holds with \( \lambda = \frac{k}{1-k} \). Then \( F \) and \( T \) has a common fuzzy fixed point in \( \overline{B}_r(x) \). That is there exists \( x^* \in \overline{B}_r(x) \) with \( \{x^*\} \subseteq Fx^* \cap Tx^* \).

**Proof.** As it is clear that

\[ d(x, [Fx]) + d(y, [Ty]) \leq \max \left( d(x, [Fx]) + d(y, [Ty]), d(x, [Fx]) + d(x, y), d(y, [Ty]) + d(x, y) \right) \]

Therefore, by Theorem 3.1, \( F \) and \( T \) has a common fuzzy fixed point in \( \overline{B}_r(x) \).

**Theorem 3.3** Let \( (X,d) \) be a complete metric space and \( x_0 \in X \). Take a mapping \( F: \overline{B}_r(x_0) \to F(X) \).

Suppose there exists a constant \( k \in (0,1/2) \) with

\[ D_l(Fx, Ty) \leq k\max \left( d(x, [Fx]) + d(y, [Ty]), d(x, [Fx]) + d(x, y), d(y, [Ty]) + d(x, y) \right) \]

for all \( x,y \in \overline{B}_r(x_0) \) and
Corollary 3.6 Let \((X,d)\) be a complete metric space. Take two mapping \(F,T: X\rightarrow F(X)\). Suppose there exists a constant \(k \in (0,1/2)\) with

\[
D_1(Fx,Ty) \leq k \left( d(x,[Fx]) + d(y,[Ty]) \right)
\]

for all \(x,y \in X\). Then \(F\) and \(T\) has a common fuzzy fixed point in \(X\). That is there exists \(x^* \in X\) with \(\{x^*\} \subseteq Fx^*\).

**Proof:** Take \(T = F\) in Theorem 3.5.

Corollary 3.8 Let \((X,d)\) be a complete metric space. Take a mapping \(F: X\rightarrow F(X)\). Suppose there exists a constant \(k \in (0,1/2)\) with

\[
D_1(Fx) \leq k \max \left( d(x,[Fx]), d(x,y) \right)
\]

for all \(x,y \in X\). Then \(F\) has a fuzzy fixed point in \(X\). That is there exists \(x^* \in X\) with \(\{x^*\} \subseteq Fx^*\).

**Proof:** Using similar argument as in Corollary 3.2 and using Theorem 3.3, we get the required result.

**CONCLUSION**

The existence of fixed point and common fixed points for fuzzy mappings satisfying the contractive conditions more general then Chatterjea type and Kannan type contractive condition on closed balls is established.

**REFERENCES**