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# On the Homology Theory of Operator Algebras

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**Abstract:** The homology (cyclic and dihedral) of unital and involuted Banach algebra are studied. The extension of results in [6] for dihedral case is studied. Also the Dihedral homolog of commutative involutive unital Banach algebra, Laurent polynomial algebra and involutive tensor Banach algebra are is studied. The main results are theorems (2,2), (3.1) (3.2), (4.1) and (4,2).

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**Key words:** Homology theory . operator algebra

## INTRODUCTION

Suppose that A is a Banach algebra and [X] be a class of polynomials defined by the norm:

$$\|[x]\| = \sup\|X(x_1, x_2, \ldots, x_n\| : \|x_i\| \le 1 \qquad (1.1)$$

Then A[x] is Banach polynomial algebra with the multiplication of coefficient of polynomials. Firstly we recall some definitions and facts about dihedral homology and its properties [1, 3]. Let A be unital Banach algebra with an involution over field K and $C_\bullet(A)$, $C_{\bullet\bullet}(A)$ bare Hochschild complex and bicomplex, respectively. The homology group of these complexes give the Simplicial (Hochschild) and cyclic homology $H_\bullet(A)$, $HC_{\bullet\bullet}(A)$ of algebra A. Consider the action of the group $\mathbb{Z}/2$ on Connes-Tsygan bicomplex $C_{\bullet\bullet}(A)$ by means of operators

$$e_r : a_0 \otimes a_1 \otimes \ldots \otimes a_n \to (-1)^{\frac{n(n-1)}{2}} e.a_0^* \otimes a_n^* \otimes \ldots \otimes a_1^*, e = \pm 1 \qquad (1.2)$$

The dihedral homology $HD_\bullet(A)$ of algebra A is defined by the hyper homology $H_\bullet(\mathbb{Z}/2; C_{\bullet\bullet}(A))$ of group $\mathbb{Z}/2$ with a coefficient in bicomplex $C_{\bullet\bullet}(A)$ [5]. Consider chain maps

$$: C_\bullet(A) \otimes C_\bullet(B) \to C_\bullet(A \otimes B) \qquad (1.3)$$

$$\perp : C_\bullet(A \otimes B) \to C_\bullet(A) \otimes C_\bullet(B) \qquad (1.4)$$

where $C_\bullet(A)$, $C_\bullet(B)$ are Hochschild complexes of algebras A and B, respectively, such that

$$e_r : a_0 \otimes a_1 \otimes \ldots \otimes a_n \to (-1)^{\frac{n(n-1)}{2}} e.a_0^* \otimes a_n^* \otimes \ldots \otimes a_1^*, e = \pm 1 \qquad (1.5)$$

$$(a \otimes a_0 \otimes \ldots \otimes a_{n-1})\ (b \otimes b_0 \otimes \ldots \otimes b_{n-1}) = \sum_s \operatorname{sgn}(s).(a \otimes b) \otimes p_{s^{-1}(0)} \otimes \ldots \otimes p_{s^{-1}(n+m-1)}$$

where the summation take over all permutation $\sigma \in \Sigma_{n+m}$ such that

$$\sigma(0) < \ldots < \sigma(n-1); \sigma(n) < \ldots < \sigma(n+m-1)$$

and

$$(p_0, \ldots, p_{n+m-1}) = (a_0 \otimes 1, \ldots, a_{n-1} \otimes 1, 1 \otimes b_0, \ldots, 1 \otimes b_{m-1})$$

$$\perp((a \otimes b) \otimes (a_0 \otimes b_0) \otimes \ldots \otimes (a_{n-1} \otimes b_{n-1})) = \sum_{i=-1}^{n-1} (aa_0 \ldots . a_i) \otimes a_{i+1} \otimes \ldots \otimes a_{i-1} \otimes (b_{i+1} \ldots . b_{n-1} b) \otimes b_0 \otimes \ldots \otimes b_i \qquad (1.6)$$

It's known that

- The chain maps $\perp$ and - induce isomorphism of Homology and the inverse is true [2]
- Clearly if A and B involution k-algebra, we can define an the involution of tensor algebra $A \otimes_k B$ as follows: $(a \otimes b)^* = a^* \otimes b^*$, $a \in A$, $b \in B$

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- The action of group $\mathbb{Z}/2$ on the Hochschild complex $C_\bullet(A)$ of algebra A by means of (1.2) induces action of $\mathbb{Z}/2$ on the Hochschild homology $HH_\bullet(A)$ of algebra A
- The spectral sequence of bicomplex $C_{\bullet\bullet}(A)$ is given by

$$\hat{E}^2_{ij}(A) = \begin{cases} HH_j(A), & i \equiv 0(\mathrm{mod}2) \\ 0, & i \equiv 1(\mathrm{mod}2) \end{cases} \tag{1.7}$$

with the differentials

$$\hat{d}^2_{ij}(A) = \begin{cases} B, & i \equiv 0(\mathrm{mod}2),\ i > 0 \\ 0, & i \equiv 1(\mathrm{mod}2),\ i = 0 \end{cases} \tag{1.8}$$

where B is Connes`s operator [5].

- The action of group $\mathbb{Z}/2$ on the spectral sequence $\hat{E}^2_{\bullet\bullet}(A)$ generates to the action of $\mathbb{Z}/2$ on every term of $\hat{E}^s_{\bullet\bullet}(A)$, $2 \prec S \le \infty$.

## POLYNOMIAL ALGEBRA AND ITS DIHEDRAL HOMOLOGY GROUP

In this part we discuss the action of $\mathbb{Z}/2$ of Hochschild homology by considering the chain operators - and $\perp$ by the following lemma

**Lemma (2.1):** Let $\alpha \in H_n(A)$ and $\beta \in H_m(B)$. Then

$$^{\varepsilon}r_n(\alpha)\top(^{\delta}r_n(\beta)) = {}^{(\varepsilon\beta)}r_{n+m}(\alpha\top\beta)$$

**Proof:** Suppose $\tilde{C}_\bullet(A)$ be normalized Hochschild complex of polynomial Banach algebra A[x]. We find the element $\perp^{(\varepsilon\beta)} r_n(\alpha\top\beta)$. Consider the representations

$$a \otimes a_0 \otimes \ldots \otimes a_{n-1} \in \alpha \text{ and } b \otimes b_0 \otimes \ldots \otimes b_{m-1} \in \beta$$

then

$$\begin{aligned} &(a \otimes a_0 \otimes \ldots \otimes a_{n-1})\top(b \otimes b_0 \otimes \ldots \otimes b_{m-1}) \\ &= \sum \mathrm{sgn}(\sigma)(a \otimes b) \otimes p_{\sigma^{-1}(0)} \otimes \ldots \otimes p_{\sigma^{-1}(n+m-1)} \xrightarrow{(\in\sigma)} r \\ &(-1)^{\frac{(n+m)(n+m+1)}{2}}(\varepsilon\delta)\sum \mathrm{sgn}(\sigma)(a \otimes b)^* \\ &\otimes p^*_{\sigma^{-1}(m+n-1)} \otimes \ldots \otimes p^*_{\sigma^{-1}(0)} \end{aligned} \tag{2.1}$$

Applying an operator $\perp$ on

$$(a \otimes a_0 \otimes \ldots \otimes a_{n-1})\top(b \otimes b_0 \otimes \ldots \otimes b_{m-1})$$

Notice that: under action of an operator $\perp$ and normalization of complexes all elements in summation $\perp(\alpha\top\beta)$ tend to zero except the following element

$$\begin{aligned} &(a \otimes b) \otimes (a_0 \otimes 1) \otimes (\otimes a_1 \otimes 1) \otimes \ldots \\ &\otimes (a_{n-1} \otimes 1) \otimes (1 \otimes b_0) \otimes (1 \otimes b_1) \otimes \ldots \otimes (1 \otimes b_{m-1}) \end{aligned}$$

Now calculating $\perp(^{(\varepsilon\beta)}r_n(\alpha\top\beta))$ :

$$\begin{aligned} &^{(\varepsilon\delta)}r((a \otimes b) \otimes (a_0 \otimes 1) \otimes \ldots \otimes (a_{n-1} \otimes 1) \otimes (1 \otimes b_0) \otimes \ldots \otimes (1 \otimes b_{m-1})) = \\ &(-1)^{\frac{(m+n)(m+n+1)}{2}}(\varepsilon\delta)(a \otimes b)^* \otimes (1 \otimes b_{m-1})^* \otimes \ldots \otimes (1 \otimes b_0)^* \otimes (a_{n-1} \otimes 1)^* \otimes \ldots \otimes (a_0 \otimes 1)^{**} \xrightarrow{\perp} \\ &(-1)^{\frac{(m+n)(m+n+1)}{2}}(\varepsilon\delta).(a^* \otimes a^*_{n-1} \otimes \ldots \otimes a^*_0) \otimes (b^* \otimes b_{m-1} \otimes \ldots \otimes b^*_n) = \\ &((-1)^{\frac{(n+1)n}{2}}\varepsilon a^* \otimes a^*_{n-1} \otimes \ldots \otimes a^*_0) \otimes ((-1)^{\frac{m(m+1)}{2}}\delta.b^* \otimes b^*_{m-1} \otimes \ldots \otimes b^*_0) = \\ &^{\varepsilon}r(a \otimes a_0 \otimes \ldots \otimes a_{n-1}) \otimes^{\delta} r(b \otimes b_0 \otimes \ldots \otimes b_{m-1}) \end{aligned}$$

Thus $^{\varepsilon}r(\alpha) \otimes (^{\delta}r(\beta)) = \perp(^{(\varepsilon\delta)}r(\alpha\top\beta)$ and hence $^{\varepsilon}r(\alpha)\top((^{\delta}r)(\beta)) = {}^{(\varepsilon\delta)}r(\alpha\top\beta)$

The main theorem of this part is the following assertion

**Theorem (2.2):** Let A be commutative involutive unital Banach algebra over K. Then

$$^{e}HD_n(A[x]) = {}^{e}HD_n(A) \oplus {}^{e}HR^{\infty}_n(A),\ e = \pm 1$$

where

$$^{e}HR^{\infty}_n(A) = {}^{e}H_n(Z/2; C_\bullet(A)) \oplus {}^{e}H_n(Z/2; C_\bullet(A)) \oplus \ldots$$

**Theorem (2.3):** If the algebra A has involution *, then the algebra A[x] also has involution i.e.

$$\left(\sum_i a_i x^i\right)^* = \sum_i a_i^* x^i,\ a_i \in A \qquad (2.2)$$

Notice that

- $A[x] = A \otimes_k k[x] : ax^i \leftrightarrow a \otimes x^i$, is involutive isomorphism (on k an involution is trivial)
- Hochshid Homology $H_\bullet(k[x])$ isomorphic to exterior algebraic form in K[x] and Connes`s operator B in Hochschild Homology corresponds to exterior differential form d [2].

Since A[x] = A⊗K[x], then

$$H_\bullet(A[x]) = H_\bullet(A) \otimes H_\bullet(k[x])$$

Under this isomorphism the Connes's operator B acts on $H_\bullet(A) \otimes H_\bullet(k[x])$ by means

$$B(\alpha \otimes \beta) = B(\alpha) \otimes \beta + (-1)^{\deg \alpha} \alpha \otimes B(\beta)$$

where

$$B(\alpha T \beta) = B(\alpha) T \beta + (-1)^{\deg \alpha} \alpha T B(\beta)$$

[2]. Since

$$H_i(k[x]) = \begin{cases} \Omega^0, & i = 0 \\ \Omega^i, & i = 1 \\ 0, & i > 1 \end{cases} \qquad (2.3)$$

where $O^i$-exterior algebraic form in K[x], then the second term in spectral sequence $\{\hat{E}^s_{\bullet\bullet} A[x]\}$ takes the form

$$\hat{E}^2_{ij}(A[x]) = \begin{cases} H_j(A) \otimes \Omega^0 \otimes H_{j-1}(A) \otimes \Omega^1, & i \equiv 0(\mathrm{mod}2) \\ 0, & i \equiv 1(\mathrm{mod}2) \end{cases} \qquad (2.4)$$

$$\hat{d}^2_{ij}(\alpha \otimes \beta) = \begin{cases} B(\alpha) \otimes \beta + (-1)^{\deg \alpha} \alpha \otimes d(\beta), & if \\ & i \equiv 0(\mathrm{mod}2),\ i > 0, \\ 0, & if \quad i \neq 0\ (\mathrm{mod}2), \end{cases} \qquad (2.5)$$

In otherwise, the cohomology of complex

$$H_0(A[x]) \xrightarrow{\hat{d}^2} \ldots \xrightarrow{\hat{d}^2} H_n(A[x]) \xrightarrow{\hat{d}^2} \ldots.$$

isomorphic to the cohomology of complex

$$H_0(A[x]) \xrightarrow{B} H_1(A[x]) \xrightarrow{B} \ldots \xrightarrow{B} H_n(A[x]) \xrightarrow{B} \ldots$$

Using Kunnth formula [2], we get:

$$H^\bullet(H_\bullet(A)) \otimes \Omega^\bullet) = H^\bullet(H_\bullet(A) \otimes H(\Omega^\bullet)) = H^\bullet(H_\bullet(A))$$

since

$$H^\bullet(\Omega^\bullet) = \begin{cases} k, & \bullet = 0 \\ 0, & \bullet > 0 \end{cases}$$

The term $\hat{E}^3_{ij}(A[x])$ has the form

$$\hat{E}^3_{ij}(A[x]) = \begin{cases} (H_j(A) \otimes \Omega^0 \otimes H_{j-1}(A) \otimes \Omega^1)\mathrm{Im}\hat{d}^2, i = 0 \\ \ker B / \mathrm{Im}B, & i \equiv 0(\mathrm{mod}2),\ i > 0 \\ 0, & i \equiv 1(\mathrm{mod}2) \end{cases} \qquad (2.6)$$

Note, that the groups $\hat{E}^3_{ij}(A[x]),\ i > 0$ isomorphic to groups $\hat{E}^3_{ij}(A),\ i > 0$

Clearly the groups $\hat{E}^3_{0j}$ isomorphic to group

$$(H_j(A)/B(H_{j-1}(A)) \oplus H_j(A) \otimes \Omega^0 / k = H^\bullet(H_\bullet(A))$$

where $\Omega^0/k$ is space of algebraic 0-form factored by k. Hence

$$\hat{E}^3_{ij}(A[x]) = \begin{cases} \hat{E}^3_{ij}(A) \oplus H_j(A) \otimes \Omega^0/k, & i = 0 \\ \hat{E}^3_{ij}(A), & i \equiv 0(\mathrm{mod}2),\ i > 0 \\ 0, & i \equiv 1(\mathrm{mod}2) \end{cases} \qquad (2.7)$$

The spectral sequence $\hat{E}^3_{0j}(A[x])$ without term

$$H^\infty_j(A) = H_j(A) \otimes \Omega^0 / k$$

tends to the cyclic homology group $HC_\bullet(A)$ and hence

$$HC_\bullet(A[x]) = HC_\bullet(A) \otimes H^\infty_\bullet(A)$$

Notice that the group $\mathbb{Z}/2$ acts on groups $E^3_{ij}(A[x]), i > 0$, by means of operators ${}^\varepsilon r$. The group $\mathbb{Z}/2$ also, acts on groups:

$$\hat{E}^3_{0j}(A[x]) = H_j(A)/B(H_{j-1}(A)) \oplus (H_j(A) \oplus \Omega^0/k) \qquad (2.8)$$

by means of

$${}^\varepsilon r(\alpha \otimes \beta \otimes w) = {}^\varepsilon r(a) \otimes {}^\varepsilon r(\beta) \otimes {}^1 r(w) \qquad (2.9)$$

Where

$$\alpha \in H_j(A)/B(H_{j-1}(A)),\ \beta \in H_j(A),\ w \in \Omega^0/k$$

Since A is Banach algebra, then the spectral sequence $\left\{{}^{\varepsilon}E^s_{\bullet\bullet}(A[x])\right\}$ expressed by

$$ {}^{\varepsilon}E^2_{pq}(A[x]) = {}^{\varepsilon}H_p(\mathbb{Z}/2, HC_q(A)) \qquad (2.10)$$

By factoring the group $HC_{\bullet}(A)\otimes H^{\infty}_{\bullet}(A)$ by the operator $(1-{}^{\varepsilon}r)$, we get the statement of theorem

Let A be commutative, involution, unital K-Banach algebra, then

$$ {}^{e}HD_i(A[x_1,\ldots,x_n]) = {}^{\varepsilon}HD_i(A) \oplus {}^{e}HR^{\infty}_i(A) \oplus \left(\bigoplus_{j=1}^{n-1} {}^{e}HR^{\infty}_i(A[x_1,\ldots,x_j])\right)$$

## DIHEDRAL HOMOLOG OF LAURENT POLYNOMIAL ALGEBRA

In this part we get the dihedral homology of Laurent polynomial algebra as follows

**Theorem (3.1):** Let A be involution Commutative unital Banach k-algebra, then

$$ {}^{e}HD_n(A[x,x^{-1}]) = {}^{e}HD_n(A) \otimes {}^{e}HD_{n-1}(A) \otimes HR^{\infty}_n(A).$$

**Proof:**

$$\left(\sum_i a_i x^i + \sum_j b_j x^{-j}\right)^* = \sum_i a_i^* x^i + \sum_j b_j^* x^{-j},\quad a_i, b_j \in A$$

Also there is an involution isomorphism k-algebra

$$ {}^{\varepsilon}HD_n(A[x,x^{-1}]) = {}^{\varepsilon}HD_n(A) \otimes {}^{\varepsilon}HD_{n-1}(A) \otimes HR^{\infty}_n(A)$$

The Hochshild homology $H_{\bullet}(k[x,x^{-1}])$ isomorphic to exterior algebraic form in $k[x,x^{-1}]$. The differential d of exterior form $\Omega^i$, $i\geq 0$ on $k[x,x^{-1}]$ corresponds to connes's operator B on Hochshild Homology $H_{\bullet}(k[x,x^{-1}])$. Since

$$A[x,x^{-1}] = \otimes_k k[x,x^{-1}]$$

then

$$H_{\bullet}(k[x,x^{-1}]) = H_{\bullet}(A) \otimes H_{\bullet}(k[x,x^{-1}])$$

since

$$B(\alpha T\beta) = B(\alpha)T\beta + (-1)^{\deg\alpha}\alpha TB(\beta)$$

then the Connes's operators B acts on $H_{\bullet}(A)\otimes H_{\bullet}(k[x,x^{-1}])$ by means of

$$B(\alpha\otimes\beta) = \beta(\alpha)\otimes\beta + (-1)^{\deg\alpha}\alpha\otimes B(\beta))$$

Since

$$H_i(k[x,x^{-1}]) = \begin{cases} \Omega^0, & i=0 \\ \Omega^i, & i=1 \\ 0, & i>1 \end{cases}$$

then the first term of spectral sequence $\left\{{}^{\varepsilon}E^s_{\bullet\bullet}(A[x,x^{-1}])\right\}$ has form

$$\hat{E}^2_{ij}(A[x,x^{-1}]) = \begin{cases} H_j(A)\otimes\Omega^0 \otimes H_{j-1}(A)\otimes\Omega^1 & \\ \qquad\qquad i\equiv 0(\mathrm{mod}2) & \\ 0, \qquad\qquad i\equiv 1(\mathrm{mod}2) & \end{cases} \qquad (3.1)$$

$$\hat{d}^2_{ij}(\alpha\otimes\beta) = \begin{cases} B(\alpha)\otimes\beta\otimes(-1)^{\deg\alpha}\alpha\otimes d(\beta),\ \text{if} \\ i\equiv 0(\mathrm{mod}2),\ i>0.,\alpha\in H_{\bullet}(A),\beta\in\Omega^{\bullet} \\ 0, \qquad\qquad \text{if}\quad i\neq 0(\mathrm{moa}2) \end{cases}$$

Calculating the cohomology of the complex

$$\begin{aligned} A[x,x^{-1}]) &\xrightarrow{\hat{d}^2} \ldots \xrightarrow{\hat{d}^2} H_n(A[x,x^{-1}]) \\ &\xrightarrow{\hat{d}^2} \ldots H^{\bullet}(H_{\bullet}(A)\Omega^{\bullet}) \end{aligned} \qquad (3.2)$$

$$\begin{aligned} &H^n(H_{\bullet}(A)\otimes\Omega^{\bullet}) \\ &= H^n(H_{\bullet}(A))\otimes\ (H^{\bullet}(\Omega^{\bullet}))_n \\ &= H^0(\Omega^{\bullet})\otimes H^n(H_{\bullet}(A))\otimes H^1(\Omega^{\bullet})\otimes H^{n-1}(H_{\bullet}(A)) \\ &= H^n(H_{\bullet}(A))\otimes H^{n-1}(H_{\bullet}(A)). \end{aligned}$$

Since

$$H^i(\Omega^{\bullet}) = \begin{cases} k, & i=0 \\ k, & i=1 \\ 0, & i>1 \end{cases}$$

the groups $\left\{{}^{\varepsilon}E^s_{\bullet\bullet}(A[x,x^{-1}])\right\}$ has the form

$$\hat{E}^3_{ij}(A[x,x^{-1}]) = \begin{cases} H_j(A)\otimes\Omega^0 H_{j-1}(A)\otimes\ldots\otimes\Omega^1/\mathrm{Im}\,\hat{d}^2,\ i=0 \\ \hat{E}^3_{ij}(A)\oplus\ \hat{E}^3_{i,j-1}(A),\ i\equiv 0(\mathrm{mod}2)\ ,i>0 \\ 0, \qquad i\equiv 1(\mathrm{mod}2). \end{cases} \qquad (3.3)$$

Clearly the group ${}^{\varepsilon}E^s_{\bullet\bullet}(A[x,x^{-1}])$ isomorphic to groups

$$(H_j(A)/B(H_{j-1}(A))\otimes(H_j(A)\otimes\Omega^0/k)\otimes(H_{j-1}(A)/B(H_{j-2}(A)))$$

In Spectral sequence $\left\{{}^{\varepsilon}E^{s}_{\bullet\bullet}(A[x,x^{-1}])\right\}$, the group ${}^{\varepsilon}E^{3}_{0j}(A[x,x^{-1}])$ without the term

$$H^{\infty}_j(A)=H_j(A)\otimes\Omega^0/k$$

converges to group $HC_{\bullet}(A)\otimes HC_{\bullet-1}(A)$ then we have

$$HC_n(A[x,x^{-1}])=HC_n(A)\otimes HC_{n-1}(A)\otimes HH^{\infty}_n(A)$$

Further more, on groups $E^3_{ij}(A[x,x^{-1}]), i>0$, the group $\mathbb{Z}/2$ acts by means of operators $({}^{\varepsilon}r\otimes{}^{\varepsilon}r_{j-1})$ and on $\widehat{E}^3_{0j}(A[x,x^{-1}])$ as follows

$${}^{\varepsilon}r(\alpha\otimes(\beta\otimes w)\otimes\delta)={}^{\varepsilon}r((\alpha)\otimes{}^{\varepsilon}r(\beta)\otimes{}^{1}r(w))\otimes{}^{\varepsilon}r(\gamma)$$

where

$$\alpha\in H_j(A)/B(H_{j-1}(A)),\quad \beta\in H_j(A)$$

$$w\in\Omega^0/k,\quad \gamma\in H_{j-1}(A)/B(H_{j-2}(A))$$

For Banach algebra the spectral sequence $\left\{{}^{\varepsilon}E^{s}_{\bullet\bullet}(A[x,x^{-1}])\right\}$ degenerated in terms

$${}^{\varepsilon}E^2_{pq}(A[x,x^{-1}])={}^{\varepsilon}H_p(\mathbb{Z}/2;HC_q(A[x,x^{-1}]))$$

Therefore factoring the group

$$HC_{\bullet}(A)\oplus HC_{\bullet-1}(A)\oplus HH^{\infty}_{\bullet}(A)$$

by the operator $(1-{}^{\varepsilon}r_{\bullet})$ the theorem is proved

In [6] Masuda has calculated the cyclic homology of algebra $A=k[x]/(x^n), n\geq 2$ with char (k)=0, we use this results for Banach algebra with an involution and study its dihedral homology.

**Theorem (3.2):** For Banach k-algebra A (where k is a field of real or complex numbers), the following dihedral homology holds for n=2:

$${}^{1}HD_i(k[x]/(x^n))=\begin{cases}k^n & , i\equiv 0(\mathrm{mod}4)\\ k & , i\equiv 2(\mathrm{mod}4)\\ 0 & , i\equiv 1,3(\mathrm{mod}4)\end{cases}$$

$${}^{-1}HD_i(k[x]/(x^n))=\begin{cases}k^{n-1} & , i\equiv 2(\mathrm{mod}4)\\ 0 & , i\neq 2(\mathrm{mod}4)\end{cases}$$

**Proof:** $H_{2m-1}(A)=k^{n-1}$ with generators $1,x,\ldots,x^{n-2}$ and $H_{2m}(A)=k^{n-1}$ with generators $x,\ldots,x^{n-1}$. Now we relate the operator ${}^{\varepsilon}r$ with the generating elements as follows

For $I=(i_1,\ldots,i_s)$ put $|I|=i+\ldots+i_s$

$$x^k=x^k\otimes_B 1_B\xrightarrow{1\otimes h_{2m}}x^k\otimes_B\sum_I(x^{m(n-1)-|I|}\otimes 1)\otimes(x\otimes x^{i_1}\otimes\ \ldots\otimes x\otimes x^{i_m})=$$

$$\sum_I x^{m(n-1)-|I|}x^k\otimes(x\otimes x^{i_1}\otimes\ \ldots\otimes x\otimes x^{i_m})\xrightarrow{{}^{\varepsilon}r}(-1)^{\frac{2m(2m+1)}{2}}\varepsilon\sum_I x^{m(n-1)-|I|}x^k\otimes x^{i_m}\otimes x\otimes\ldots\otimes x^{i_1}\otimes x)$$

$$=(-1)^m\varepsilon\sum_I x^{m(n-1)-|I|}x^k\otimes_B(1_B\otimes x^{i_m}\otimes x\otimes\ldots\otimes x^{i_1}\otimes x)\xrightarrow{1\otimes h_{2m}}(-1)^m\varepsilon x^k\otimes_B(1_B)=(-1)^m\varepsilon x^k$$

Thus on the Hochchinld homology $H_{2m}(A)$ the operator ${}^{\varepsilon}r_{2m}$ acts with multiplication on $(-1)^m\varepsilon$. Using the spectral sequence of bicomplex $C_{\bullet\bullet}(A)$ we have

$$\widehat{E}^2_{ij}(A)=\begin{cases}k^n, & i\equiv 0(\mathrm{mod}2),\quad j=0\\ k^{n-1}, & i\equiv 2(\mathrm{mod}2), i>0\\ 0, & i\equiv 1(\mathrm{mod}2).\end{cases}$$

In [6] Massuda has shown that $B:H_{2m-1}(A)\to H_{2m}(A)$ its zero homomorphism and $B:H_{2m}(A)\to H_{2m+1}(A)$ is an isomrphism but $B:H_0(A)\to H_1(A)$ is epimorphism. In $\widehat{E}^3_{\bullet\bullet}(A)$ the spectral sequence $\widehat{E}^3_{ij}(A)$ is given by

$$\widehat{E}^2_{ij}(A) = \begin{cases} k^n, & i = j = 0 \\ k^{n-1}, & i = 0,\ j \equiv 0(\mathrm{mod}2) \\ 0, & j = 0,\ i \equiv 0(\mathrm{mod}2) \end{cases}$$

Since A is Banach k-algebra (char(k)=0), then there is a spectral sequence of second term. Now by factoring the group

$$HC_\bullet(A) = \bigoplus_{\bullet = p+q} \widehat{E}^3_{ij}(A)$$

by $\mathrm{Im}(1-{}^{\varepsilon}r_\bullet)$ we get

$$\widehat{E}^3_{ij}(A)/(1-{}^{1}r_j) = \begin{cases} k^n, & i = j = 0 \\ k, & j = 0, i \equiv 0(\mathrm{mod}2), i > 0 \\ k^{n-1}, & i = 0, j \equiv 0(\mathrm{mod}4), j > 0 \end{cases}$$

$$\widehat{E}^3_{ij}(A)/(1-{}^{(-1)}r_j) = \begin{cases} k^{n-1}, & i = 0, j \equiv 2(\mathrm{mod}4), j > 0 \\ 0, & i > 0,\ j \neq 2(\mathrm{mod}4) \end{cases}$$

Thus we have

$${}^{1}HD_i(k[x]/(x^n)) = \begin{cases} k^n, & i \equiv 0(\mathrm{mod}4) \\ k, & i \equiv 2(\mathrm{mod}4) \\ 0, & i \equiv 1;\ 3(\mathrm{mod}4) \end{cases}$$

$${}^{-1}HD_i(k[x]/(x^n)) = \begin{cases} k^{n-1}, & i \equiv 2(\mathrm{mod}4) \\ 0, & i \neq 2(\mathrm{mod}4) \end{cases}$$

## DIHEDRAL HOMOLOGY OF TENSOR ALGEBRA

Let A involution k-Banach algebra, where field k (k is a real or complex nunmber). Consider tensor algebra $T_A(A)$ with involutive A-bimodule M. Recall that A-bimodule M has involution, if there is an automorphism $*:M \to M$ of order two such that $(amb)^* = b^* m^* a^*$, $a,b \in A$. Suppose that $P_\bullet$-chain complex of A-bimodules with an involution. Consider the complex

$$S^{(k)}(A, P_\bullet) = A \otimes_{A \otimes A^{op}} P_\bullet^{\otimes(k+1)}$$

Acting on $S^{(k)}(A,P_\bullet)$ by two automorphisms

$$t_k(P_0 \otimes \ldots \otimes p_k) = (-1)^s P_k \otimes P_0 \otimes \ldots \otimes P_{k-1} \qquad (4.1)$$

$${}^{\varepsilon}\theta_k(P_0 \otimes \ldots \otimes p_k) = (-1)^{\upsilon} \varepsilon P_k^* \otimes \ldots \otimes P_1^* \otimes P_0^* \qquad (4.2)$$

Where

$$s = (\deg P_k)\left(\sum_{i=o}^{k-1} \deg P_i\right)$$

$$\upsilon = \left(k + \sum_{i=o}^{k} \deg P_i\right)\left(k + \sum_{i=o}^{k} \deg P_i - 1\right)/2$$

The automorphism s $t_k$ and ${}^{\varepsilon}\theta_k$ represent group dihedral $D_{k+1}$ of order 2(k+1). If $P_\bullet$ be free involution A-bimodule of resolution involution A-bimodule M, then the complex $S^{(k)}(A,P_\bullet)$ can be written as follows $S^{(k)}(A,M)$.

**Theorem (4.1):** Let M-involutive A-bimodule and

$$\mathrm{Tor}_i^A(M,M) = 0, i > 0$$

Then

$${}^{\varepsilon}HD_i(T_A(A)) = {}^{\varepsilon}HD_i(A) \oplus (\bigoplus_{k=0}^{\infty}(D_{k+1}S^{(k)}(A,M)).$$

**Proof:** The long exact sequence of relative dihedral homology [4] for sequence of involutive homeomorphism algebra $A \xrightarrow{i} T_A(M) \to 0$, is given by

$$\ldots \to {}^{\varepsilon}HD_i(A \xrightarrow{i} T_A(M)) \xrightarrow{0} {}^{\varepsilon}HD_i(A \to 0) \to$$
$$\to {}^{\varepsilon}HD_i(T_A(M) \to 0) \to {}^{\varepsilon}HD_{i-1}(A \xrightarrow{i} T_A(M)) \xrightarrow{0} \ldots$$

Since A is the direct sum in $T_A(M)$, then the long exact sequence splits and hence

$${}^{\varepsilon}HD_i(T_A(M)) = {}^{\varepsilon}HD_i(A) \oplus HD_i(A \xrightarrow{i} T_A(M))$$

we show that

$${}^{\varepsilon}HD_i(A \to T_A(M)) = \bigoplus_{k=0}^{\infty} {}^{\varepsilon}H_i(D_{k+1}, S^k(A,M))$$

Really

$$T_A(P_\bullet)/(A + [T_A(P_\bullet), T_A(P_\bullet)] + \mathrm{Im}(1 - {}^{\varepsilon}r))$$
$$= \bigoplus_{k=0}^{\infty} P_\bullet^{\otimes(k+1)} / ([T_A(P_\bullet), T_A(P_\bullet)] + \mathrm{Im}(1 - {}^{\varepsilon}r))$$

Since

$${}^{\varepsilon}r(P_0 \otimes \ldots \otimes P_k) = (-1)^{\upsilon'} \varepsilon (P_0 \otimes \ldots \otimes P_k)^*$$
$$= (-1)^{\upsilon'} \varepsilon P_k^* \otimes \ldots \otimes P_1^* \otimes P_0^*$$

Where

$$\upsilon' = (k + \deg(P_0 \otimes P_1 \otimes \ldots \otimes P_k))(k + \deg(P_0 \otimes \ldots \otimes P_k) - 1)/2 = \upsilon$$

then, we have an isomorphism

$$\bigoplus_{k=0}^{\infty} (P_\bullet^{\otimes(k+1)})/([T_A(P_\bullet), T_A(P_\bullet)] + \mathrm{Im}(1 - {}^{\varepsilon}r))$$
$$= \bigoplus_{k=0}^{\infty} A_{A \otimes A^{op}} P_\bullet^{\otimes(k+1)} / (\mathrm{Im}(1 - t_k) + \mathrm{Im}(1 - {}^{\varepsilon}\theta_k))$$

The homology of chain complex

$$\otimes_{k=0}^{\infty} A \otimes_{A \otimes A^{op}} P_\bullet^{\otimes(k+1)} / (\mathrm{Im}(1 - t_k) + \mathrm{Im}(1 - {}^{\varepsilon}\theta_k))$$

in exactly coincide with $\bigoplus_{k=0}^{\infty} {}^{\varepsilon}H_\bullet(D_{k+1}, S^{(k)}(A,M))$.

If A is augmented k-algebra with involution then,

$${}^{\varepsilon}\overline{HD_\bullet}(A) = \ker({}^{\varepsilon}HD_\bullet(A) \to {}^{\varepsilon}HD_\bullet(k) \qquad (4.3)$$

**Corollary (4.2):** Let V-Banach space with an involution

$${}^{\varepsilon}\overline{HD_\bullet}(T_k(V)) = \bigoplus_{k=0}^{\infty} {}^{\varepsilon}H_\bullet(D_{k+1}, V^{\otimes(k+1)})$$

**Proof:**

$${}^{\varepsilon}HD_\bullet(T_k(V)) = {}^{\varepsilon}HD_\bullet(k) \oplus (\bigoplus_{k=0}^{\infty} {}^{\varepsilon}H_\bullet(D_{i+1}, S^{(i)}(k,v)))$$

and following isomorphism

$$\bigoplus_{k=0}^{\infty} {}^{\varepsilon}H_\bullet(D_{k+1}, S^{(k)}(k,v)) \cong \bigoplus_{k=0}^{\infty} {}^{\varepsilon}H_\bullet(D_{k+1}, V^{\otimes(k+1)}) \quad (4.4)$$

Really $S_\bullet^{(i)}(k,v) = R_\bullet^{\otimes(i+1)}$, where $R_\bullet$ is free k-module resolution of k-module V. We can get an isomorphism (4.4) by considering the sequence $V \leftarrow 0 \leftarrow 0 \leftarrow \ldots$ from the resolution $R_\bullet$. The corollary is proved.

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