World Applied Sciences Journal 25 (12): 1789-1796, 2013
ISSN 1818-4952

DOI: 10.5829/idosi.wasj.2013.25.12.72

Polynomial Approximation Method for Solving Composite Fractional Relaxation/Oscillation Equations

K. Krishnaveni, K. Kannan and S. Raja Balachandar

Department of Mathematics, School of Humanities and Sciences, SASTRA University, Thanjavur-613401, Tamilnadu, India

Submitted: Mar 8, 2013; **Accepted:** Aug 2, 2013; **Published:** Nov 8, 2013

Abstract: In this study, a polynomial approximation method based on the shifted Legendre polynomials for some linear differential equations of fractional order with initial values namely fractional oscillation equation, fractional relaxation equation is proposed. The properties of shifted Legendre polynomials together with the Caputo fractional derivative are used to reduce the problem to the solution of linear algebraic equations. The illustrative examples demonstrate its applicability, validity and simplicity of the approximation scheme.

Key words: Fractional differential equation . shifted legendre polynomials method . integral equations . fractional oscillation equation . fractional relaxation equation

INTRODUCTION

Many physical and engineering phenomena have been governed by fractional differential equations [30]. Many authors have demonstrated the applications of fractional calculus in the motion of a rigid plate immersed in a Newtonian fluid [15, 19] namely composite fractional oscillation equation, unsteady motion of a particle accelerates in a viscous field due to the gravity force [2], namely composite fractional relaxation equation.

The other field includes: theory of stellar structure, the thermal behaviour of spherical cloud of gas, isothermal gas sphere, theory of currents [1], damping behavior of many viscoelastic materials [3], fluid-dynamic traffic model [6] and electrochemistry [29]. In the past decades, researchers have devoted considerable effort to find robust and stable numerical and analytical solve for FDE, governing such type of systems.

In general, there exists no method that yields an exact solution for FDE. Only approximate solutions can be derived using linearization or perturbation methods. These methods include the homotopy perturbation method [25-28], Adomian's decomposition method [16, 23], homotopy analysis method [14, 41], variational iteration method [24], generalized differential transform method [12, 28] and a few more methods [3, 9-11, 20, 40] that provide convergence solutions for particular problems.

Recently, operational matrix based wavelets are found in literature in obtaining fast solutions to many fields of science and engineering [33]. The main advantage of the wavelet technique is its ability to transform complex problems into a system of algebraic equations. Wavelets are being applied to a wide and growing range of applications such as signal processing, data and image compression and statistics [4, 7]. The application of Legendre wavelets for solving differential and integral equations is thoroughly considered in [1, 8, 13, 17, 18, 21, 22, 31, 32, 38-40] and references therein.

The main characteristic behind operational matrix based wavelets approach using this technique is that it reduces these problems to those of solving a system of linear /nonlinear algebraic equations thus greatly simplifying the problem. In this Legendre wavelets method a truncated orthogonal series is used for numerical integration of differential equations, with a goal of obtaining efficient computational solutions.

The various applications of shifted Legendre polynomials and theoretical analysis are studied by many researchers and several papers [34-38] have appeared in the literature concerned.

In this paper we intend to extend the application of Shifted Legendre polynomials method (SLPM) to find the approximate solution of composite fractional relaxation/oscillation equations. The general form of linear fractional differential equation is

Corresponding Author: K. Krishnaveni, Department of Mathematics, School of Humanities and Sciences, SASTRA University, Thanjavur-613401, Tamilnadu, India.

$$D^n y(t) - a D_*^{\alpha} y(t) - b y(t) - f(t) = 0$$
$$t > 0, n-1 < \alpha \le n, n = 1,2$$

with initial conditions

$$y^{(j)}(0) = s_j,\ j = 0,1,\dots,n-1$$

where s_j are constants and y(t) is assumed to be a casual function of time, i.e., vanishing for t<0. If f(t)=0, (1) is known as composite fractional relaxation equation corresponds to the Basset Problem [2]. If f(t)=8, the above problem is known as composite fractional oscillation equation [19].

Recently Odibat and Momani [25] implemented the variational iteration method, adomian decomposition method and fractional difference method to solve the composite fractional oscillation equation and composite fractional relaxation equation.

DEFINITIONS AND PRELIMINARIES

The fractional calculus is a name for the theory of integrals and derivatives of arbitrary order, which unifies and generalizes the notions of integer-order differentiation and n-fold integration [8, 18]. Various definitions of fractional integration and differentiation have been proposed in the last two centuries. These include Grunwald_Letnikov, Riemann_Liouville, Weyl, Reiz, Caputo fractional operators [29, 30]. The Riemann_Liouville derivative has certain disadvantages when trying to model real-world phenomena with fractional differential equations. Therefore, we have used a modified fractional differential operator D^{α} proposed by Caputo [3, 5].

Definition 2.1: A real function *f* (*x*), *x* ? 0, is said to be in the space C_μ, $\mu \in R$ if there exists a real number *p* ? ? ? such that $f(x) = x^p f_1(x)$, where $f_1(x)$? *C*[0,?]. Clearly $C_\mu < C_\beta$ if $\beta < \mu$.

Definition 2.2: A function f(x), x>0, is said to be in the space $C_\mu^m, m \in N \cup \{0\}$ iff $f^{(m)} \in C_\mu$.

Definition 2.3: The Riemann-Liouville fractional integral operator of order α≥0, of a function, $f \in C_\mu, \mu \ge -1$
is defined as

$$J^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt,\ \alpha > 0, x > 0$$
$$J^0 f(x) = f(x)$$

Some of the properties of the operator J^{α} are for $f \in C_\mu$, $\mu \ge -1$, $\alpha,\beta \ge 0$ and γ>-1

$$J^{\alpha} J^{\beta} f(x) = J^{\alpha+\beta} f(x)$$

$$J^{\alpha} J^{\beta} f(x) = J^{\beta} J^{\alpha} f(x)$$

$$J^{\alpha} x^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}$$

Definition 2.4: The left sided Riemann_Liouville fractional differential equation of order q≥0, of a function $f \in C_q$, $q \ge -1$, is defined as:

$$ {}_aD_t^{q} f(t) = \begin{cases} \dfrac{1}{\Gamma(-q)} \displaystyle\int_a^t (t-\tau)^{-q-1} f(\tau)\, d\tau & q < 0 \\ f(t) & q = 0 \\ D^n \left[{}_aD_t^{q-n} f(t) \right] & q > 0 \end{cases}$$

where n is the smallest integer larger than q, i.e., n-1≤q<n and Γ is the Gamma function:

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt$$

Definition 2.5: The fractional derivative of f(x) in the Caputo sense is defined as

$$D_*^{\alpha} f(x) = J^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t)dt$$

for

$$m-1 < \alpha \le m, m \in N, x > 0, f \in C_{-1}^m$$

$$D^{\alpha} C = 0, \quad \text{(Cisaconsant)}$$

$$D^{\alpha} x^{\beta} = \begin{cases} 0, & \text{for } \beta \in N_0 \text{ and } \beta < \lceil \alpha \rceil, \\ \dfrac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} x^{\beta-\alpha}, & \text{for } \beta \in N_0 \\ & \text{and } \beta \ge \lceil \alpha \rceil \text{ or } \beta \notin N \text{ and } \beta > \lfloor \alpha \rfloor \end{cases}$$

where α>0 is the order of the derivative and n is the smallest integer greater than α and we use the ceiling function $\lceil \alpha \rceil$ to denote the smallest integer greater than or equal to α and the floor function $\lfloor \alpha \rfloor$ to denote the largest integer less than or equal to α.

Lemma 2.1: If $m-1 < \alpha \le m, m \in N$ and $f \in C_\mu^m$, $m \ge -1$, then

$$D_*^{\alpha} J^{\alpha} f(x) = f(x)$$

$$J^{\alpha} D_*^{q} f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+)\frac{x^k}{k!},\ x > 0$$

To obtain a numerical scheme for the approximation of Caputo derivative, we can use a representation that has been introduced by Elliots [11];

$$D_*^{q} f(x) = \frac{1}{\Gamma(-q)}\int_0^x \frac{f(s)-f(0)}{(x-s)^{1+q}} ds, 0<q<1$$

where the integral in the above equation is a Hadamard finite-part integral.

Definition 2.6: The Grunwald-Letnikov fractional differential equation operator of order q:

$$_aD_t^{q} f(t) = \frac{d^q f(t)}{d(t-a)^q} = \lim_{N\to\infty}\left[\frac{t-a}{N}\right]^{-q}\sum_{j=0}^{N-1}(-1)^j f\left(t - j\left[\frac{t-a}{N}\right]\right)$$

For a wide class of functions, the Grunwald-Letnikov and the Riemann-Liouville definitions are equivalent.

SHIFTED LEGENDRE POLYNOMIALS AND ITS PROPERTIES

The well-known Legendre polynomials are defined on the interval [-1,1] and can be determined with the aid of the following recurrence formulae:

$$L_{i+1}(z) = \frac{2i+1}{i+1} z L_i(z) - \frac{i}{i+1} L_{i-1}(z),\ i = 1,2,...$$

where $L_0(z) = 1$ and $L_0(z) = z$.

In order to use these polynomials on the interval $x \in [0,1]$, Saadatmandi and Dehghan [38] utilized the so-called shifted Legendre polynomials by introducing the change of variable $z = 2x-1$.

Let the shifted Legendre polynomials $L_i(2x-1)$ be denoted by $P_i(x)$. Then $P_i(x)$ can be obtained as follows:

$$P_{i+1(x)} = \frac{(2i+1)(2x-1)}{(i+1)} P_i(x) - \frac{i}{i+1} P_{i-1}(x), i = 1,2,... \quad (2)$$

where $P_0(x) = 1$ and $P_i(x) = 2x-1$. The analytic form of the shifted Legendre polynomial $P_i(x)$ of degree i given by

$$P_i(x) = \sum_{k=0}^{i} (-1)^{i+k} \frac{(i+k)!}{(i-k)!} \frac{x^k}{(k!)^2} \quad (3)$$

Note that $P_i(x) = (-1)^i$ and $P_i(1) = 1$. The orthogonality condition is

$$\int_0^1 P_i(x) P_j(x) dx = \begin{cases} \frac{1}{2i+1} & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases} \quad (4)$$

A function y(x), square integrable in [0,1], may be expressed in terms of shifted Legendre polynomials as

$$y(x) = \sum_{j=0}^{\infty} c_j P_j(x)$$

where the coefficients c_j are given by

$$c_j = (2j+1)\int_0^1 y(x) P_j(x) dx, j = 1,2,....$$

In practice, only the first (m+1) terms of shifted Legendre polynomials are considered. Then we have

$$y(x) = \sum_{j=0}^{m} c_j P_j(x) = C^T \Phi(x) \quad (5)$$

where the shifted Legendre coefficient vector C and the shifted Legendre vector $\Phi(x)$ are given by $C^T = [c_0,\ldots,c_m]$,

$$\Phi(x) = [P_0(x), P_1(x), \ldots, p_m(x)]^T \quad (6)$$

Operational matrices

a. Operational matrix for derivative

The derivative of the vector $\Phi(x)$ can be expressed by

$$\frac{d\Phi(x)}{dx} = D^{(1)}\Phi(x) \quad (7)$$

where $D^{(1)}$ is the $(m+1)\times(m+1)$ operational matrix of derivative given by

$$D^{(1)} = (d_{ij}) = \begin{cases} 2(2j+1), & \text{for } j = i-k, \begin{cases} k = 1,3,...,m, & \text{if } m \text{ odd} \\ k = 1,3,...,m-1, & \text{if } m \text{ even} \end{cases} \\ 0, & \text{otherwise} \end{cases}$$

By using Eq.(1), it is clear that

$$\frac{d^n\Phi(x)}{dx^n} = \left(D^{(1)}\right)^n \Phi(x)$$

where $n \in N$ and the superscript, in $D^{(1)}$, denotes matrix powers. Thus

$$D^{(n)} = \left(D^{(1)}\right)^n, \quad n = 1,2,.... \tag{8}$$

b. Operational matrix for fractional derivative

Lemma 3.1: Let $P_i(x)$ be a shifted Legendre polynomial then

$$D^{\alpha}P_i(x) = 0, \quad i = 0,1,...,\lceil \alpha \rceil - 1, \alpha > 0 \tag{9}$$

In the following theorem, we generalize the operational matrix of derivative of shifted Legendre polynomials given in (2) for fractional derivative.

Theorem 31: Let $\Phi(x)$ be shifted Legendre vector defined in (1) and also suppose α>0 then

$$D^{\alpha}\Phi(x) \cong D^{(\alpha)}\Phi(x) \tag{10}$$

where $D^{(\alpha)}$ is the $(m+1)\times(m+1)$ operational matrix of fractional derivative of order α in the Caputo sense and is defined as follows:

$$D^{(\alpha)} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \\ \sum_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil} \theta_{\lceil\alpha\rceil,0,k} & \sum_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil} \theta_{\lceil\alpha\rceil,1,k} & \cdots & \sum_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil} \theta_{\lceil\alpha\rceil,m,k} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{k=\lceil\alpha\rceil}^{i} \theta_{i,0,k} & \sum_{k=\lceil\alpha\rceil}^{i} \theta_{i,1,k} & \cdots & \sum_{k=\lceil\alpha\rceil}^{i} \theta_{i,m,k} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{k=\lceil\alpha\rceil}^{m} \theta_{m,0,k} & \sum_{k=\lceil\alpha\rceil}^{m} \theta_{m,1,k} & \cdots & \sum_{k=\lceil\alpha\rceil}^{m} \theta_{m,m,k} \end{pmatrix} \tag{11}$$

where $\theta_{i,j,k}$ is given by

$$\theta_{i,j,k} = (2j+1)\sum_{l=0}^{j} \frac{(-1)^{i+j+k+l}(i+k)!(l+j)!}{(i-k)!\,k!\,\Gamma(k-\alpha+1)(j-l)!(l!)^2(k+l-\alpha+1)} \tag{12}$$

Note that in $D^{(\alpha)}$, the first $\lceil\alpha\rceil$ rows, are all zero. The proof of the lemma 4.1 and theorem 4.1 are available in [38].

SLPM FOR LINEAR FRACTIONAL DIFFERENTIAL EQUATION

To solve problem (1), we approximate y(x) by the shifted Legendre polynomial as

$$y(x) = \sum_{j=0}^{m} c_j P_j(x) = C^T\Phi(x)$$

$$f(x) = \sum_{j=0}^{m} e_j P_j(x) = E^T\Phi(x)$$

Equation (1) becomes

$$C^T D^n \Phi(x) - aC^T D^{\alpha}\Phi(x) - bC^T\Phi(x) - E^T\Phi(x) = 0$$

By tau method [35], we generate m-1 linear equations by using

$$\int_0^1 R_m(x)P_j(x)dx = 0, \; j = 0,1,...,m-1$$

where

$$R_m(x) = C^T D^n \Phi(x) - aC^T D^{\alpha}\Phi(x) - bC^T\Phi(x) - E^T\Phi(x)$$

Also, by using the initial conditions and the definition of $D^{(n)}$, we get n linear equations

$$y^{(j)}(0) = C^T D^{(j)}\Phi(0) = s_j$$

These linear equations can be solved for unknown coefficients of the vector C. Consequently, y(x) can be evaluated.

APPLICATION OF OPERATIONAL MATRIX FOR LINEAR FRACTIONAL DIFFERENTIAL EQUATION

Example 1: Consider the composite fractional oscillation equation [26]

$$\frac{dy}{dt} - aD_*^{\alpha}y(t) - by(t) = 0, \, t > 0, 0 < \alpha \le 1 \tag{13}$$

subject to the initial conditions y(0)=1 and assume that a = b = -1.

By applying SLPM for α = 0.5 with m=5, the approximate solution of equation (13) is

$$y(x) = \sum_{j=0}^{5} c_j P_j(x) = C^T\Phi(x)$$

Here we have

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 & 0 \\ 2 & 0 & 10 & 0 & 0 & 0 \\ 0 & 6 & 0 & 14 & 0 & 0 \\ 2 & 0 & 10 & 0 & 18 & 0 \end{pmatrix}$$

Table 1: Numerical values when α = 0.25, 0.5, 0.75 and a = b = -1 for example 1

	α = 0.25			α = 0.5			α = 0.75		
t	ADM	VIM	SLPM	ADM	VIM	SLPM	ADM	VIM	SLPM
0.0	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
0.1	0.751577	0.751577	0.751577	0.662104	0.662104	0.662104	0.566982	0.566982	0.566982
0.2	0.605730	0.605730	0.605730	0.543694	0.543694	0.543694	0.493983	0.493983	0.493983
0.3	0.499314	0.499314	0.499314	0.463863	0.463863	0.463863	0.443495	0.443495	0.443495
0.4	0.417812	0.417812	0.417812	0.404072	0.404072	0.404072	0.403821	0.403821	0.403821
0.5	0.353711	0.353711	0.353711	0.356911	0.356911	0.356911	0.370859	0.370859	0.370859
0.6	0.302381	0.302381	0.302381	0.318509	0.318509	0.318509	0.342608	0.342608	0.342608
0.7	0.260714	0.260714	0.260714	0.286543	0.286543	0.286543	0.317909	0.317909	0.317909
0.8	0.226518	0.226518	0.226518	0.259495	0.259495	0.259495	0.296016	0.296016	0.296016
0.9	0.198191	0.198191	0.198191	0.236315	0.236315	0.236315	0.276413	0.276413	0.276413
1.0	0.174535	0.174535	0.174535	0.216243	0.216243	0.216243	0.258722	0.258722	0.258722

$$D^{0.5} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1.5045 & 0.9027 & -0.2149 & 0.1003 & -0.0586 & 0.0386 \\ -0.9027 & 1.9344 & 1.5045 & -0.4103 & 0.2083 & -0.1290 \\ 1.2896 & -0.6018 & 2.2274 & 1.9906 & -0.5862 & 0.3132 \\ -1.0030 & 1.5240 & -0.3858 & 2.4619 & 2.4099 & -0.7466 \\ 1.2310 & -0.8101 & 1.6999 & -0.2110 & 2.6625 & 2.7844 \end{pmatrix}$$

We obtain the following values of c_j:

$$c_0 = 0.1742,\ c_1 = -0.7240,\ c_2 = -0.1790$$
$$c_3 = -0.1525,\ c_4 = 0.0268,\ c_5 = -0.0263$$

Thus we can write the approximate solution is

$$y(x) = -6.62\, t^5 + 18.4277\, t^4 - 21.5159\, t^3 + 11.4311\, t^2 - 3.5282\, t + 0.9247$$

For α = 0.25, the coefficients of Legendre polynomial are

$$c_0 = 0.4578,\ c_1 = -0.2689,\ c_2 = 0.1880$$
$$c_3 = -0.0376,\ c_4 = 0.0193,\ c_5 = -0.0108$$

and the approximate solution of (e-1) is $y(x)=-2.7243\, t^5+8.16\, t^4-9.5043\, t^3+6.2609\, t^2-2.8269\, t+0.9824$.

Similarly For α = 0.75; c_0 = -1.1113, c_1= -2.3316, c_2 = -0.6480, c_3 = -0.1594, c_4 = 0.0379, c_5 = -0.0386 and $y(x) = -9.7366\, t^5+26.9936\, t^4-30.1289\, t^3+12.4173\, t^2-4.6046\, t+0.8081$. For α = 1; c_0 = -3.3264, c_1 = -3.0946, c_2 = -0.84175, c_3 = -0.076148, c_4 = -0.0026764, c_5 = -0.000032937 and the approximate solution is

$$y(x) = -0.0083\, t^5 - 0.1666\, t^{4-} 1.1667\, t^3 - 3\, t^{2-} 2\, t - 1$$

Next we compare SLPM solution with ADM and VIM, the numerical values of y(t) when α = 0.25, 0.5, 0.75 are presented in Table 1.

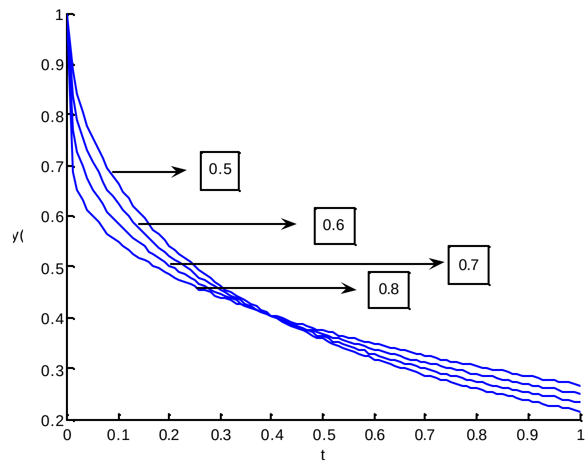

Fig. 1: Comparison of y(x) for M = 10 and with α = 0.5, 0.6, 0.7, 0.8 for example 1

From the Table 1 we observe that SLPM provides the same solution similar to ADM and VIM but the computational complexity of SLPM is less than the other two because ADM and VIM are required more time to find the coefficients of fractional polynomial terms.

The numerical values of y(t) with α = 0.5 to 0.8 are plotted in Fig. 1 for M = 10. Figure 1 also shows that the convergence of the SLPM for different values of α.

Example 2: Consider the composite fractional oscillation equation [26]

$$\frac{d^2y}{dt^2} - aD_*^{\alpha} y(t) - by(t) - 8 = 0, t \geq 0, n-1 < \alpha \leq n, n = 1,2 \quad (14)$$

subject to the initial conditions $y(0) = 0$, $y^1(0) = 0$.

Table 2: Numerical values when α = 0.5, 1.0, 1.5 and a = b = -1 for example 2

	α = 0.5			α = 1.0			α = 1.5		
T	ADM	VIM	SLPM	ADM	VIM	SLPM	ADM	VIM	SLPM
0.0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
0.1	0.039874	0.039874	0.039874	0.038925	0.038925	0.038925	0.036478	0.036478	0.036478
0.2	0.158512	0.158512	0.158512	0.153742	0.153742	0.153742	0.140640	0.140640	0.140640
0.3	0.353625	0.353625	0.353625	0.340182	0.340182	0.340182	0.307485	0.307485	0.307485
0.4	0.622083	0.622083	0.622083	0.593846	0.593846	0.593846	0.533284	0.533284	0.533284
0.5	0.960047	0.960047	0.960047	0.910214	0.910214	0.910214	0.814757	0.814757	0.814757
0.6	1.363093	1.363093	1.363093	1.284685	1.284685	1.284685	1.148840	1.148840	1.148840
0.7	1.826257	1.826257	1.826257	1.712597	1.712597	1.712597	1.532571	1.532571	1.532571
0.8	2.344224	2.344224	2.344224	2.189258	2.189258	2.189258	1.963033	1.963033	1.963033
0.9	2.911278	2.911278	2.911278	2.709964	2.709964	2.709964	2.437331	2.437331	2.437331
1.0	3.521462	3.521462	3.521462	3.270029	3.270029	3.270029	2.952567	2.952567	2.952567

By applying SLPM with m = 12, the approximate solution of (14) is

$$y(x) = \sum_{j=0}^{12} c_j P_j(x) = C^T \Phi(x)$$

For $\alpha = 0.25$, we obtain the following values of c_j:

$c_0 = 1.1758$, $c_1 = 1.6916$, $c_2 = 0.45761$
$c_3 = -0.064886$, $c_4 = -0.0059663$, $c_5 = 0.00072406$
$c_6 = 4.4372e\text{-}006$, $c_7 = 1.3503e\text{-}006$
$c_8 = -1.2431e\text{-}006$, $c_9 = 4.3956e\text{-}007$, $c_{10} = -1.7426e\text{-}007$
$c_{11} = 7.9906e\text{-}008$, $c_{12} = -3.8184e\text{-}008$

Thus we can write the approximate solution is $y(x) = -0.10326\, t^{12} + 0.6759\, t^{11} - 1.9719\, t^{10} + 3.3897\, t^9 - 3.8328\, t^8 + 3.0193\, t^7 - 1.7064\, t^6 + 0.87336\, t^5 - 1.0651\, t^4 - 0.024536\, t^3 + 4.0006\, t^2 - 7.3427e\text{-}006\, t + 1.6448e\text{-}008$.

Similarly for $\alpha = 0.5$; $c_0 = 1.1338$, $c_1 = 1.618$, $c_2 = 0.4211$, $c_3 = -0.067924$, $c_4 = -0.0038585$, $c_5 = 0.0008574$, $c_6 = -4.4366e\text{-}005$, $c_7 = 7.8602e\text{-}006$, $c_8 = -3.285e\text{-}006$, $c_9 = 1.2713e\text{-}006$, $c_{10} = -5.3933e\text{-}007$, $c_{11} = 2.6145e\text{-}007$, $c_{12} = -1.3099e\text{-}007$ and the approximate solution is $y(x) = -0.35422\, t^{12} + 2.3098\, t^{11} - 6.7045\, t^{10} + 11.446\, t^9 - 12.811\, t^8 + 9.9516\, t^7 - 5.5949\, t^6 + 2.605\, t^5 - 1.6183\, t^4 - 0.12942\, t^3 + 4.0023\, t^2 - 2.8364e\text{-}005 t + 6.1069e\text{-}008$.

For $\alpha = 0.75$; $c_0 = 1.0784$, $c_1 = 1.5264$, $c_2 = 0.38326$, $c_3 = -0.066491$, $c_4 = -0.00096567$, $c_5 = 0.00069175$, $c_6 = -7.3762e\text{-}005$, $c_7 = 1.3626e\text{-}005$, $c_8 = -4.8706e\text{-}006$, $c_9 = 1.952e\text{-}006$, $c_{10} = -8.7265e\text{-}007$, $c_{11} = 4.4487e\text{-}007$, $c_{12} = -2.3274e\text{-}007$ and $y(x) = -0.62936\, t^{12} + 4.09\, t^{11} - 11.82\, t^{10} + 20.061\, t^9 - 22.269\, t^8 + 17.098\, t^7 - 9.4654\, t^6 + 4.1476\, t^5 - 1.8313\, t^4 - 0.46505\, t^3 + 4.0049\, t^2 - 5.6906e\text{-}005\, t + 1.1711e\text{-}007$.

Next we compare SLPM solution with ADM and VIM, the numerical values of y(t) when α = 0.5, 1.0, 1.5 are presented in Table 2. From the Table 2 we observe that SLPM provides the same solution similar to ADM and VIM but the computational complexity of SLPM is less than the other two because ADM and VIM are required more time to find the coefficients of fractional polynomial terms. The numerical values of y(t) with α = 1.1 to 1.5 are plotted in Fig. 2 for m = 12. Figure 2 also shows that the convergence of the SLPM for different values of α.

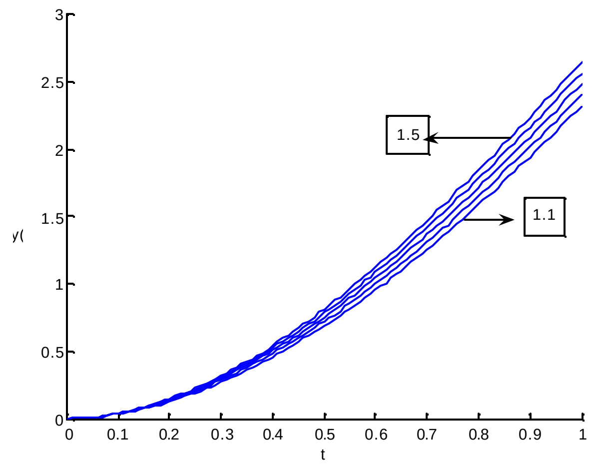

Fig. 2: Comparison of y(x) for m = 12 and with α = 1.1, 1.2, 1.3, 1.4, 1.5 for example 2

FEATURES OF SLPM

The proposed method is very simple in application and is more accurate in comparison with other methods mentioned above (VIM and ADM). By using SLPM the solution can be obtained in bigger interval. Unlike ADM and VIM, the SLPM does not need the Adomian polynomials, Lagrange multiplier, correction functional, stationary conditions and calculating integrals, which eliminate the complications that exist in the ADM and VIM. Comparison of SLPM with ADM and VIM reveals that numerical results of all the methods when applied to example 1 and example 2are nearly the same some tolerance, the approximation obtained by SLPM converge faster than ADM and VIM.

CONCLUSION

In this work, we proposed the Shifted Legendre Polynomials Method (SLPM) for solving composite fractional relaxation/oscillation equations. The properties of shifted Legendre polynomials and Caputo derivative are used to reduce the problem to the solution of linear algebraic equations with appropriate coefficients which provide exact solutions for all the chosen problems. It is clearly reveal that validity and potential use of applicability to any phenomena governed by this equation. In future, we may use this proposed method for solving other non linear fractional integro-differential equations and fractional partial differential equations also.

REFERENCES

1. Ali Yousefi, S., 2006. Legendre wavelets method for solving differential equations of Lane-Emden type. Appl. Math. Comp., 181:417-1422. http://dx.doi.org/10.1016/j.amc.2006.02.031.
2. Basset, A.B., 1910. On the descent of a sphere in a viscous liquid. Quart. J. Math., 42: 369-381.
3. Bagley, R.L. and P.J. Torvik, 1983. Fractional calculus: A different approach to the analysis of viscoelastically damped structures, AIAA J., 21 (5): 741-748.
4. Beylkin, G., R. Coifman and V. Rokhlin, 1991. Fast wavelet transforms and numerical algorithms. I, Commun. Pure Appl. Math., 44: 141-183.
5. Caputo, M., 1967. Linear models of dissipation whose Q is almost frequency independent. Part II, J. Roy. Astr. Soc., 13: 529-539.
6. Chechkin, A.V., R. Gorenflo, I.M. Sokolov and V. Gonchar, Yu, 2003. Distributed order time fractional diffusion equation. Frac. Calc. Appl. Anal., 6: 259-279.
7. Chui, C.K., 1997. Wavelets: A Mathematical Tool for Signal Analysis. SIAM, Philadelphia, PA,
8. Das, S., 2008. Functional Fractional Calculus for System Identification and Controls, Springer, New York.
9. Diethelm, K., 1997. An algorithm for the numerical solution of differential equations of fractional order. Electronic Trans. Num. Ana., 5: 1-6.
10 Diethelm, K., N.J. Ford and A.D. Freed, 2002. A predictor corrector approach for the numerical solution of fractional differential equation. Non Lin. Dyn., 29: 3-22.
11. Elliott, D., 1993. An asymptotic analysis of two algorithms for certain Hadamard finitepart integrals, IMA J. Numer. Anal., 13: 445-462.
12. Erturk, V., S. Momani and Z. Odibat, 2008. Application of generalized differential transform method to multi-order fractional differential equations. Comm. Nonlin. Sci. Numer. Simul, 13 (8): 1642-1654. http://dx.doi.org/10.1016/j.cnsns.2007.02.006.
13. Gu, J.S. and W.S. Jiang, 1996. The Haar wavelets operational matrix of integration. Int. J. Syst. Sci., 27: 623-628.
14. Hashim, I., O. Abdulaziz and S. Momani, 2009. Homotopy analysis method for fractional IVPs. Commun Nonlin. Sci. Num. Simul., 14: 674-84. http://dx.doi.org/10.1016/j.cnsns.2007.09.014.
15. Hilfer, R., 2000. Applications of Fractional Calculus in Physics. World Scientific Publishing Company.
16. Jafari H. and V.D. Gejji, 2006. Solving a system of nonlinear fractional differential equations using adomain decomposition. Appl. Math. Comput., 196 (2): 644-651. http://dx.doi.org/10.1016/j.cam.2005.10.017.
17. Khellat, F. and S.A. Yousefi, 2006. The linear Legendre mother wavelets operational matrix of integration and its application. J. Frank Inst., 343: 181-190. http://dx.doi.org/10.1016/j.jfranklin.2005.11.002.
18. Kilbas, A.A., H.M. Srivastava and J.J. Trujillo, 2006. Theory and Applications of Fractional Differential Equations, Elsevier, San Diego.
19. Kulish, V.V. and L. Lage José, 2002. Application of fractional calculus to fluid mechanics. J. Fluids Eng., 124: 3(803)(4 pages) doi:10.1115/1.1478062.
20. Lepik, Ü., 2009. Solving fractional integral equations by the Haar wavelet method. Appl. Math. Comp., 214: 468-78. http://dx.doi.org/10.1016/j.amc.2009.04.015.
21. Liu, N. and Lin En-Bing, 1998. Legendre wavelet method for numerical solutions of partial differential equations. SIAM J. Math. Anal., 29: 1040-1065. doi: 10.1002/num.20417.
22. Maleknejad, K. and S. Sohrabi, 2007. Numerical solution of Fredholm integral equations of the first kind by using Legendre wavelets. Appl. Math. Comput., 186: 836-843. http://dx.doi.org/10.1016/j.amc.2006.08.023.

23. Momani, S. and Al-Khaled, 2005. Numerical solutions for systems of fractional differential equations by the decomposition method. Appl. Math. Comput., 162 (3): 1351-1365. http://dx.doi.org/10.1016/j.amc.2004.03.014.
24. Momani, S. and Z. Odibat, 2006. Analytical approach to linear fractional partial differential equations arising in fluid mechanics. Physics Letters A, 355 (4-5): 271-279. http://dx.doi.org/10.1016/j.physleta.2006.02.048.
25. Momani, S. and Z. Odibat, 2007a. Homotopy perturbation method for nonlinear partial differential equations of fractional order. Phys. Lett. A 2007; (5-6): 345-350. http://dx.doi.org/10.1016/j.physleta.2007.01.046.
26. Momani, S. and Z. Odibat, 2007b. Numerical comparison of methods for solving linear differential equations of fractional order. Chaos, Solitons and Fractals, 31: 1248-1255. http://dx.doi.org/10.1016/j.chaos.2005.10.068.
27. Momani, S. and Z. Odibat, 2008a. Modified homotopy perturbation method: application to quadratic riccati differential equation of fractional order. Chaos Soli. Fract, 36 (1): 167-174. http://dx.doi.org/10.1016/j.chaos.2006.06.041.
28. Momani, S. and Z. Odibat, 2008b. A Generalized differential transform method for linear partial differential equations of fractional order. Appl. Math. Lett., 21 (2): 194-199. http://dx.doi.org/10.1016/j.aml.2007.02.022.
29. Oldham, K.B., 2010. Fractional differential equations in electrochemistry. Adv Eng Soft, 41: 9-12. http://dx.doi.org/10.1016/j.cnsns.2011.01.014.
30. Podlubny, I., 1999. Fractional Differential Equations. San Diego: Academic Press.
31. Razzaghi, M. and S. Yousefi, 2000. Legendre wavelets direct method for variational problems. Math. Comput. Simul., 53: 185-192. http://dx.doi.org/10.1016/S0378-4754(00)00170-1.
32. Razzaghi, M. and S. Yousefi, 2001. The Legendre wavelets operational matrix of integration. Int. J. Syst. Sci., 32: 495-502. doi:10.1080/00207720120227.
33. Resnikoff, H.L. and R.O.Jr. Wells, 1998. Wavelet Analysis. New York, Springer-Verlag.
34. Saadatmandi, A., M. Razzaghi and M. Dehghan, 2005. Hartley series approximations for the parabolic equations. Intern. J. Comput. Math., 82: 1149-1156. doi:10.1080/00207160500113066.
35. Saadatmandi, A. and M. Dehghan, 2006. A Tau method for the one-dimensional parabolic inverse problem subject to temperature overspecification. Comput.Math. Appl, 52: 933-940. doi:10.1016/j.camwa.2006.04.017.
36. Saadatmandi, A. and M. Dehghan, 2007. Numerical solution of the one-dimensional wave equation with an integral condition. Numer. Methods Partial Differential Equations, 23: 282-292. doi: 10.1002/num.20177.
37. Saadatmandi, A. and M. Dehghan, 2008. Numerical solution of a mathematical model for capillary formation in tumor angiogenesis via the tau method. Commun. Numer. Methods. Eng., 24: 467-1474. doi: 10.1002/cnm.1045.
38. Saadatmandi, A. and M. Dehghan, 2010. A new operational matrix matrix for solving fractional-order differential equations. Computers and Mathematics with Applications, 59: 1326-1336. http://dx.doi.org/10.1016/j.camwa.2009.07.006.
39. Samko, S.G., A.A. Kilbas and O.I. Marichev, 1993. Fractional Integrals and Derivatives. Gorden and Breach Science Publishers, Yverdon and New York.
40. Zheng, X. and X. Yang, 2009. Techniques for solving integral and differential equations by Legendre wavelets. Int. J. Syst. Sci., 40: 1127-1137.
41. Zurigat, M., S. Momani, Z. Odibat and A. Alawneh, 2010. The homotopy analysis method for handling systems of fractional differential equations. Appl. Math. Model, 34 (1): 24-35.