World Applied Sciences Journal 25 (12): 1783-1788, 2013
ISSN 1818-4952

DOI: 10.5829/idosi.wasj.2013.25.12.13515

Mathematical Simulation of a Bend of Polygonal Plates having Complex Outlines in a Plan by Means of Collocation Method

Oleg Vladimirovich Mashkin

Ural Federal University named after the first President of Russia B.N.Yeltsin, Ekaterinburg, Russia

Submitted: Oct 8, 2013; **Accepted:** Nov 2, 2013; **Published:** Nov 8, 2013

Abstract: The article is devoted to the issue of mathematical simulation and bending of polygonal isotropic plates with use of method of Overdetermined Boundary Collocation Method (OBCM). Methods and the results of calculation made by Overdetermined boundary collocation method of evenly loaded polygonal plates having constant thickness are given.

Key words: Polygonal plate . Bend . Overdetermined boundary collocation method

INTRODUCTION

Polygonal isotropic plates are often used as elements of construction in buildings and facilities and are applied in other industries as well: for example in machine-building, aircraft engineering etc.

To do mathematical simulation of polygonal plates different approximate ways of solving boundary-value problems are used. Approximate ways of calculation are divided into 3 big groups: projection, grid-based, projection-and grid-based methods. Projection methods are Ritz, Bubnov-Galerkin and Trefftz methods [1] methods of least squares, collocations; grid-based methods are finite-difference methods and variation-difference methods; projection-and grid-based are different variants of finite element method.

Mathematical simulation is not restricted to obtaining simultaneous differential equations which describe stressed deformed state of plates but mathematical tools using which it is possible to integrate these equations with given preciseness.

Boundary collocation method intended for solving of a linear problem of bending of square fixed along perimeter evenly loaded plate subject to concentrated force applied to the center was initially used by J. Barta in 1937 [2].

Solution of an equation describing plate's bending [3] is expressed as sum of general (in Clebsch's form) and particular (when the way of applied load is taken into consideration) solutions.

Availability of multiple symmetry has simplified calculations and allowed to get satisfying results when boundary conditions are fulfilled in 2 and 3 points situated on 1/8 part of plate's perimeter. It is obvious that absence of symmetry would make calculations more complex; the resulting problem can be solved by quickly acting computers and appropriate algorithm.

In the beginning of sixties H.D. Conway had published articles [4, 5] in which boundary problems of bending, as well as problems of sustainability and fluctuations of square, triangle and having symmetrical polygonal form plate were solved by boundary collocation method; this method also solved simple problems of twisting of rods.

It ignited interest of scientists in BCM and A.W. Leissa, F.W. Niedenfuhr, C.Lo and W.E. Clausen published a number of articles in American magazine 'Rocket machinery and astrounautics', Robinson-in other scientific editions in the USA [6].

In one of their works A.W. Leissa and W.E. Clausen had applied method of re-defined boundary collocation, probably for the first time, having formed 38 equations for definition of 20 coefficients of Clebsch series, it means that they satisfied 19 boundary conditions in 19 different points of plate profile. Overdetermined simultaneous linear equations were solved by the authors by least squares method.

Russian scientists had shown interest in BCM as well [6] solving different particular problems of bending and sustainability of plates and shells but they did not develop program complexes on the base of BCM or Overdetermined Boundary Collocation Method (OBCM).

In the beginning of 70s after initiative of professor M. Kornishin [7, 8] active development of collocation methods of solving linear and geometrically non-linear boundary problems connected with mechanics of plates and slightly slanted shells was started. They proposed and realized methods of orthogonal, overdetermined internal and overdetermined boundary collocation.

Corresponding Author: Mashkin, Ural Federal University named after the first President of Russia B.N.Yeltsin, ul. Mira, 19, 620002, Ekaterinburg, Russia

Based on OBCM the program for calculation of convex polygonal plates [9, 10] was developed and many applied problems ordered by enterprises were solved.

In this article OBCM was taken to do mathematical simulation of stressed and deformed state of polygonal in plan plates with constant thickness which are being fixed along perimeter.

It was shown that when OBCM is used the way of positioning of points of collocation does not greatly influence the preciseness and repeatability of results.

OBCM

Balance of simply connected thin resilient plates of any form in a plan at little deflexion is described by well-known equation of Germain-Lagrange which can be written in polar coordinates system [3].

$$\Delta\Delta w = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}\right) = \frac{q}{D} \quad (1)$$

Here $w(r,\theta)$ is the function of deflexion of a plate; r,θ are polar coordinates connected with Cartesian system of coordinates by ratio

$$r^2 = \sqrt{x^2 + y^2},\ \theta = \operatorname{arctg}(y/x)$$

q is evenly distributed cross-sectional load,

$$D = Eh^3/(12(1-\nu^2))$$

is cylindrical stiffness; E, v-elasticity module and Poisson coefficient of isotropic material.

When we assume that the plate is fixed along its perimeter, we can substitute boundary conditions in the form of:

$$w\big|_\Gamma = 0, \qquad \frac{\partial w}{\partial n}\bigg|_\Gamma = 0 \quad (2)$$

where n-normal to appropriate side of fixed perimeter, which forms angle a with axis x.

Let us present approximate value of boundary problem in Clebsch's form [3]-it as a sum of general solution of homogeneous and particular solution of non-homogeneous equations (1)

$$w = w_0 + w_q = \sum_{i=1}^{s}\left(A_i r^{j-1}\cos(j-1)\theta + A_{s+i} r^{j+1}\cos(j-1)\theta + A_{2s+i} r^j \sin j\theta + A_{3s+i} r^{j+2}\sin j\theta\right) + Cqr^4 \quad (3)$$

where w_0-general solution of homogeneous equation (1) in Clebsch's form;
w_q-particular solution of equation (1) with

$$C = 3(1-\nu^2)/(16Eh^3)$$

Solution (3) turns the balance equation into truth therefore the constants of the solution. A_i must be defined from boundary conditions (2), which are true discretely in collocation points.

Substituting (3) and necessary derivatives into (2) and assuming that mis-ties in collocation points with coordinates r_γ, θ_γ $(\gamma = 1 \ldots S_\Gamma)$, must be equal to zero we shall get:

$$\sum_{i=1}^{s}\left(\sum_{k=1}^{4} A_{s(k-1)+i}\Psi_{ik}\right) + Cqr^4\Bigg|_{\substack{r=r_\gamma \\ \theta=\theta_\gamma}} = 0 \quad (4)$$

$$\sum_{i=1}^{s}\left(\sum_{k=1}^{4} A_{s(k-1)+i}\Psi_{i,k+4}\right) + Cq4r^3\cos(\theta-\alpha)\Bigg|_{\substack{r=r_\gamma \\ \theta=\theta_\gamma \\ \alpha=\alpha_\gamma}} = 0 \quad (5)$$

Here

$$\Psi_{i1} = r^{j-1}\cos(j-1)\theta;\quad \Psi_{i2} = \Psi_{i1}r^2;\quad \Psi_{i3} = r^j \sin j\theta$$
$$\Psi_{i4} = \Psi_{i3}r^2;\quad \Psi_{i5} = (j-1)r^{j-2}\cos[(j-2)\theta+\alpha]$$
$$\Psi_{i6} = r^j\{j\cos[(j-2)\theta+\alpha] + \cos(j\theta-\alpha)\}$$
$$\Psi_{i7} = j\,r^{j-1}\sin[(j-1)\theta+\alpha]$$
$$\Psi_{i8} = r^{j+1}\{(j+1)\sin[(j-1)\theta+\alpha] + \sin[(j+1)\theta-\alpha]\}$$

Now we can form overdetermined simultaneous linear equations $(n = 4s)\times(N = 2s_\Gamma)$, where s_Γ = number of points on the perimeter of plate; 4s=the number of coefficients in the system (4) and (5).

Solving simultaneous equations using LU-analysis, we can find coefficients A_l and calculate by formulas [3] values which characterize SDS (stressed-deformed state) of the plate.

Deflexions can be found using formula (3), bending moments-using formulas:

$$M_r = -D\left(\frac{\partial^2 w}{\partial r^2} + \nu\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}\right)\right)$$

$$M_t = -D\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2} + \nu\frac{\partial^2 w}{\partial r^2}\right)$$

$$M_{rt} = (1-\nu)D\left(\frac{1}{r}\frac{\partial^2 w}{\partial r\,\partial\theta} - \frac{1}{r^2}\frac{\partial w}{\partial\theta}\right) \quad (6)$$

and normal stress-using the following expressions:

$$\sigma_x = M_x/W, \sigma_y = M_y/W; W = h^2/6 \quad (7)$$

Allocation of points on the boundary of areas: While solving linear (non-linear) boundary problems BCM and OBCM equations which are true in the area of G can be satisfied precisely (approximately) and the constants of solution can be defined from boundary conditions on the boundary of "Γ" area, which are true discretely in collocation points and therefore we have the issue of allocation of points on the boundary between the areas.

Figure 1 demonstrates a fragment of area in Cartesian coordinates which is restricted by straight line segments. Let us consider a variant of even allocation of n points on the segment B_i-B_{i+1}. We assume that the first and the last points are 1 step away from points B_i and B_{i+1}.
Then:

$$x_\gamma = x_i + \frac{x_{i+1} - x_i}{n+1} \cdot \gamma, y_\gamma = y_i + \frac{y_{i+1} - y_i}{n+1} \cdot \gamma, \gamma = 1 \ldots n \quad (8)$$

If the first and the last collocation points are 1/2 of step away from angular points then:

$$x_\gamma = x_i + \frac{x_{i+1} - x_i}{2n} \cdot (2\gamma - 1), \quad y_\gamma = y_i + \frac{y_{i+1} - y_i}{2n} \cdot (2\gamma - 1), \gamma = 1 \ldots n \quad (9)$$

Variant is possible when angular points B_i and B_{i+1} are collocation points.
In this case

$$x_\gamma = x_i + \frac{x_{i+1} - x_i}{n-1} \cdot (\gamma - 1), \quad y_\gamma = y_i + \frac{y_{i+1} - y_i}{n-1} \cdot (\gamma - 1), \gamma = 1 \ldots n \quad (10)$$

but it is necessary to be accurate in developing appropriate simultaneous equations on the boundary in order that boundary condition (for example, absence of deflexion) was not fulfilled twice for the angular point.

As detailed investigations have shown even (uniform) allocations of points should be used for solving boundary problems by OBCM. In those cases when solution is fulfilled by BCM (for example, for rectangular areas) collocation points should be allocated in Chebyshev's way, condensing them towards angular points.

Figure 2 demonstrates how you should define coordinates of points in Chebyshev's way. Forming

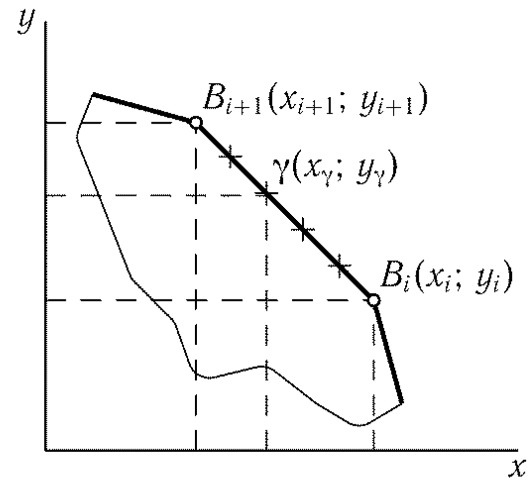

Fig. 1: Even allocation of points on the boundary of the area

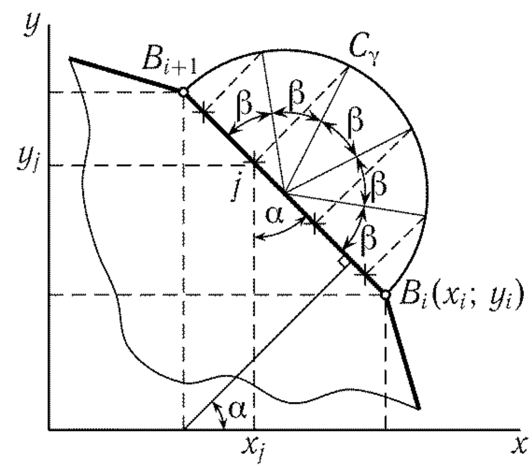

Fig. 2: Allocation of points on the boundary of the area with condensing to angular points

perpendicular lines from C_γ points to the segment B_i-B_{i+1} (diameter of semicircle) gives:

$$x_j = x_i - 0.5 s_i (1 - \cos\frac{\pi\gamma}{n+1}) \sin\alpha$$
$$y_j = y_i - 0.5 s_i (1 - \cos\frac{\pi\gamma}{n+1}) \cos\alpha \quad (11)$$
$$j = \gamma = 1 \ldots n, s_i = \sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2}$$

Repeatability and accuracy of obtained results: Main parameters of OBCM-n, N, N/n are found in the process of solution of boundary problem by means of mis-tie analysis in fulfillment of boundary conditions on the outer perimeter and repeatability of values which characterize SDS of plates.

Table 1: Repeatability of solution results for hexagon plate

N/n	s	w(0;0)	$M_x(0;0)$	$M_y(0;0)$	N/n	s	w(0;0)	$M_x(0;0)$	$M_y(0;0)$
Allocation of collocation points which are 1 step away from angular points									
24/12	3	6,15778	1.13450	1.13401	168/84	21	6.50441	1.16352	1.16302
72/36	9	6,80434	1.19226	1.19183	192/96	24	6.50440	1.16351	1.16301
120/60	15	6,51597	1.16444	1.16394	216/108	27	6.50317	1.16342	1.16292
144/72	18	6.51515	1.16431	1.16382	240/120	30	6.50328	1.16343	1.16293
Allocation of collocation points which are 1/2 step away from angular points									
24/12	3	6,71985	1.18515	1.18463	168/84	21	6.50374	1.16342	1.16292
72/36	9	6,88354	1.19331	1.19301	192/96	24	6.50432	1.16345	1.16295
120/60	15	6,51154	1.16376	1.16326	216/108	27	6.50312	1.16339	1.16290
144/72	18	6.51194	1.16379	1.16329	240/120	30	6.50335	1.16340	1.16290
Allocation of collocation points in Chebyshev's way									
24/12	3	6,71985	1.18515	1.18463	168/84	21	6.50538	1.16323	1.16274
72/36	9	6,95177	1.19308	1.19270	192/96	24	6.50550	1.16321	1.16272
120/60	15	6,50921	1.16332	1.16282	216/108	27	6.50257	1.16326	1.16276
144/72	18	6.51004	1.16335	1.16285	240/120	30	6.50254	1.16325	1.16276

To evaluate repeatability of results of boundary problem solution we shall take the state of fixed symmetrical hexagon plate subject to evenly distributed load q = 1000 kPa with the following input data: E = $2.0\cdot10^8$ kPa, v = 0.3, side a = 0.134 m, h = 0.003 M-thickness of the plate.

Detailed calculations have proved that OBCM is more effective with plates of complex form than BCM and with two-fold overidentification of SLAEs (simultaneous linear algebra equations) OBCM solution are reliably repeated with increment in n and N and mis-ties of solution quickly decrease locating near angles on the sides of external supporting perimeter. The results of calculations on hexagon plate are given in Table 1.

These results must be compared with analytical solution [8] and solutions obtained by MKE in MicroFe [9] and Lira program complexes.

Analytical value of deflexion and bending moments in the center of the plate can be calculated using formulas [8]:

$$w(0;0) = 0.009979\frac{qa^4}{D} = 6.5056\text{mm}$$

$$M_x(0;0) = M_y(0;0) = 0.049835(1+\nu)qa^2 = 1.16329\text{kNm/m}$$

Value of deflexion and bending moments in the center of the plate calculated by program complex MicroFe [9]:

$$w(0;0) = 6.47\text{mm}\ ,\quad M_x(0;0) = M_y(0;0) = 1.15\text{kN m/m}$$

and plate area is broken down into 316 finite elements.

Value of deflexion and bending moments in the center of the plate.....calculated by Program complex Lira 9.6 R9:

$$w(0;0) = 6.46184\text{mm}$$

$$M_x(0;0) = 1.159\ \text{kNm/m},\ M_y(0;0) = 1.161\text{kNm/m}$$

And plate area is automatically broken down into 493 rectangular elements (simultaneous equations order 1260).

Data analysis of contents of Table 1 shows that values of deflexion and bending moments in the center of plate are reliably repeated and while allocating 10 points (N/n = 120/60) on every side the results are enough precise; they practically are not different from precise.

Some numeral results of solution of boundary problems: Let us do calculations on evenly loaded hexagon fixed along its perimeter reinforced concrete plate which is symmetrical with respect to axis x (Fig. 4). Calculations were done as requested by some building company in accordance with following input data:

$$q = 2\text{kN/m}^2;\ a = 2\text{m};\ b = 3\text{m};$$
$$h = 0{,}1\text{m};\ E = 2\cdot10^7\ \text{kN/m}^2;\ \nu = 0{,}2$$

In solution (3) 48 collocations points are allocated along the perimeter with distance of 1 step from angular points; the degree of overidentification of simultaneous linear algebraic equations is 2. The results of solution are given in line #1 of the Table 2.

Table 2: Results of solution of hexagon reinforced concrete plate

?	w(0;0)	$M_x(0;0)$	$M_y(0;0)$	$M_y(0;3)$	$M_x(-2;0)$	$M_x(2;0)$
1	0.59534	1.07251	0.56470	-1.57689	-2.29801	-2.0562
2	0.59234	1.06716	0.54545	-1.28900	-1.95000	-1.8180

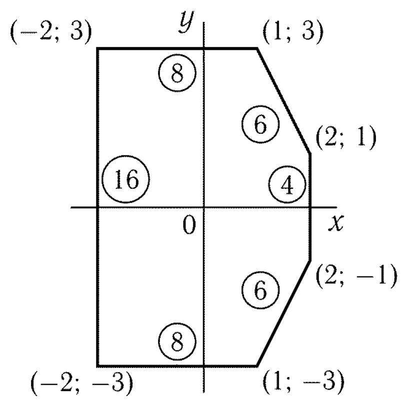

Fig. 4: Geometry of hexagon reinforced concrete plate

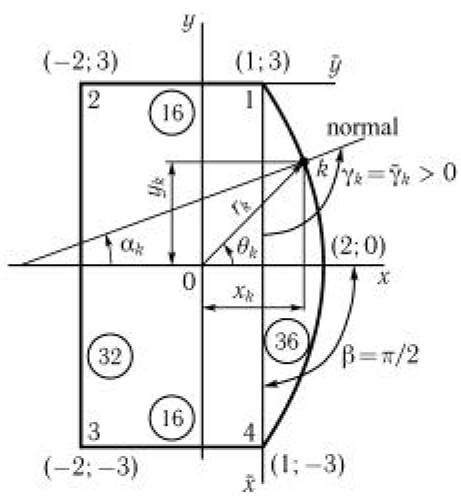

Fig. 5: Plate's geometry

For assumed parameters the mis-tie solutions on deflexions and turns on the plate are extremely small and further increment in n and N will not improve calculations' results.

Line#2 of the Table 2 shows the solution obtained by use of program complex Lira 9.6 R6, where area of the plate is automatically broken down into 1100 triangular elements (simultaneous equations order 1515).

Use of OBCM allowed to decrease simultaneous equations order 16 times, in comparison with MKE.

Figure 5 demonstrates calculation results on reinforced concrete plate where one of fixed edges has parabolic outline. Figure 6 shows plate's geometry in meters.

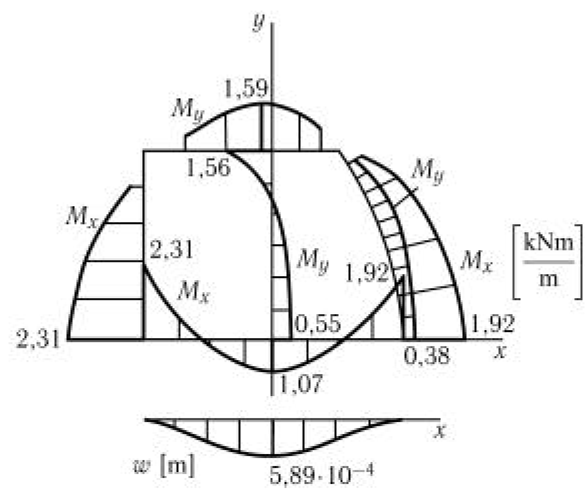

Fig. 6: Deflexions and bending moments in the plate

In local coordinates system $\tilde{x}$, $\tilde{y}$ parabola equation has a form of- $\tilde{y} = \tilde{x}(6-\tilde{x})/9$ and therefore the coordinates of the point k are as follows:

$$\tilde{x}_k = 6k/(m_4+1),\quad \tilde{y}_k = \tilde{x}_k(6-\tilde{x}_k)/9,\quad k = 1\ldots m_4$$

where m_4-is a number of points on curved side.

Coordinates x, y of k in general coordinates system can be found using formulas:

$$\begin{aligned} x_k &= x_{i+1} + \tilde{x}_k\cos\beta + \tilde{y}_k\sin\beta, \\ y_k &= y_{i+1} - \tilde{x}_k\sin\beta + \tilde{y}_k\cos\beta, \end{aligned} \quad (???\quad \beta = \pi/2)\ \ldots$$

where $\beta = \pi/2$

where $\tilde{x}_k$, $\tilde{y}_k$ are coordinates of k in local coordinates system;

β = betwee x and $\tilde{x}$ axis.

Value of angle a_k can be found from expression

$$\alpha_k = \gamma_k - \beta \quad \gamma_k = \tilde{\gamma}_k$$

if $\tilde{\gamma}_k > 0$; $\gamma_k = \tilde{\gamma}_k + \pi$, if $\tilde{\gamma}_k < 0$;

$$\tilde{\gamma}_k = \mathrm{arctg}(-1/\tilde{y}_{k'})$$

$$\tilde{y}' = d\tilde{y}/d\tilde{x} = (6-2\tilde{x})/9$$

$$\theta_k = \mathrm{arctg}(y_k/x_k)$$

$$r_k = \sqrt{x_k^2 + y_k^2}$$

Value of angle α_k can be found from expression $\alpha_k = \gamma_k - \beta$, when $\gamma_k = \tilde{\gamma}_k$, if $\tilde{\gamma}_k > 0$; $\gamma_k = \tilde{\gamma}_k + \pi$, if $\tilde{\gamma}_k < 0$;

$$\tilde{\gamma}_k = \text{arctg}(-1/\tilde{y}_{k'})$$

$$\tilde{y}' = d\tilde{y}/d\tilde{x} = (6 - 2\tilde{x})/9$$

$$\theta_k = \text{arctg}(y_k/x_k)$$

$$r_k = \sqrt{x_k^2 + y_k^2}$$

The calculations were done taking into account the symmetry with regard to x axis with the following input data: a = 2m; b = 3m; h = 0.1m; E = $2 \cdot 10^7$ kN/m^2; v = 0,2; q = 2kN/m^2; s = 25, N/n = 100/50. Points along the perimeter are allocated with 1 step distance from the angles. Number of points is shown in Fig. 5 (circled numbers). The analysis has demonstrated that accuracy of the results is very high.

Inference: Developed on the base of OBCM program enables to simulate SDS of polygonal evenly loaded isotropic plates very effectively and with high accuracy. The programs are implemented using MATLAB language applied in modern personal computers.

Obvious advantage of OBCM is that while solving boundary-value problems in linear form balance equation of the plates is true and the constants of solutions can be found from discretely satisfied boundary conditions on outer perimeter of the plate.

REFERENES

1. Li, Z.-C., T.-T. Lu, H.-Y. Hu and A.H.-D. Cheng, 2008. Trefftz and Collocation Methods. Boston: WIT Press.
2. Barta, J., Uber die naherungsweise Losung einiger zweidimensionaler Elastizitataufgaben (periodics) ZAMM, pp: 184-185.
3. Timoshenko, S. and S. Woinowsky-Krieger, 1959. Theory of plates and shells. New York: McGraw-Hill.
4. Conway, H., 1960.The approximate Analysis of Certain Boundary-Value Problems. J. of Appl. Mech., Trans. ASME, 27: 82.
5. Conway, H., 1961. The Bending, Buckling and Flexural Vibration of simply supported polygonal plates by Point-Matching. Trans. ASME, S.E., Vol: 28 (2).
6. Rogalevitch, V., 1980. Solutions to boundary problems in theory of plates and shells by collocation method. Robustness and sustainability of shells. Kazan: Publishing house of KFTI AN.
7. Kornishin, M., 1964. Non-linear problems of plates and slightly slanted shells theory and solution methods. Moscow: Nauka.
8. Kornishin, M. and V. Rogalevitch, 1975. Application of collocation method for solution of linear and geometrically non-linear problems of bending of plates and shells. 10th Pan-Union Conference on shell and plate theory, Tbilisi: Metsniereba, pp: 345-354.
9. Mashkin, O. and V. Rogalevitch, 2011. Calculations of plates by boundary collocation methods. Ekaterinburg: Publishing house AMB, pp: 76.
10. Buksha, V., O. Mashkin and V. Rogalevitch, 2007. Calculations on plates and shells by collocation methods. Ekaterinburg: AMB.
11. Vainberg, D., 1973. Reference-book on robustness, sustainability and fluctuations of plates. Kiev: Budivelnik.
12. Semenov, V., 2009. Verification Report on Program complex MicroFe. Moscow: Publishing house of Russian Academy of Architecture and building sciences.