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New Nonlinear Fractional Model for Differential Equations

A. Golmamadian
Faculty of Mathematical Sciences, University of Gonbad, Iran

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Abstract: In this present work, the fractional derivatives in the sense of modified Riemann-Liouville derivative and the direct algebraic method are employed for constructing the exact complex solutions of nonlinear time-fractional partial differential equations. The power of this manageable method is presented by applying it to several examples.

Key words: Direct · Fractional equation · Fractional Sharma-Tasso-Olever equation.

INTRODUCTION

The investigation of exact solutions to nonlinear Fractional differential equations plays an important role in various applications in physics, biology, engineering, signal processing, systems identification, control theory, finance and fractional dynamics [1-3]. Recently, a large amount of literature has been provided to construct the solutions of fractional ordinary differential equations, integral equations and fractional partial differential equations of physical interest. Several powerful methods have been proposed to obtain approximate and exact solutions of fractional differential equations, such as the Adomian decomposition method [4,5], the variational iteration method [6-8], the homotopy analysis method [9-11], the homotopy perturbation method [12-14], the Lagrange characteristic method [15], the fractional sub-equation method [16, 21, 22] and so on.

In [17], Jumarie proposed a modified Riemann-Liouville derivative. With this kind of fractional derivative and some useful formulas, we can convert fractional differential equations into integer-order differential equations by variable transformation. The direct algebraic method [18-20] can be used to construct the exact solutions for some time fractional differential equations. The present paper investigates for the first time the applicability and effectiveness of the first integral method on fractional nonlinear partial differential equations.

The Modified Riemann-Liouville Derivative and the Direct Algebraic Method: In this section, we first give some definitions and properties of the modified Riemann-Liouville derivative which are used further in this paper.

Assume that $f:\mathbb{R}\to\mathbb{R}, x\to f(x)$ denote a continuous (but not necessarily differentiable) function. The Jumarie modified Riemann-Liouville derivative of order α is defined by the expression:

$$D_x^{\alpha} f(x) = \begin{cases} \dfrac{1}{\Gamma(-\alpha)}\displaystyle\int_0^x (x-\xi)^{-\alpha-1}\left[f(\xi)-f(0)\right]d\xi & \alpha \prec 0 \\ \dfrac{1}{\Gamma(1-\alpha)}\dfrac{d}{dx}\displaystyle\int_0^x (x-\xi)^{-\alpha}\left[f(\xi)-f(0)\right]d\xi & 0\prec\alpha\prec 1 \\ (f^{(n)}(x))^{(\alpha-n)} & n\le\alpha\le n+1, n\ge 1 \end{cases} \tag{1}$$

Some properties of the fractional modified Riemann-Liouville derivative were summarized and three useful formulas of them are:

$$D_x^{\alpha} x^{\gamma} = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)} x^{\gamma-\alpha}, \gamma \succ 0 \tag{2}$$

$$D_x^{\alpha}(u(x)v(x)) = v(x)D_x^{\alpha}u(x) + u(x)D_x^{\alpha}v(x), \tag{3}$$

$$D_x^{\alpha}\left[f(u(x))\right] = f'(u)D_x^{\alpha}u(x) = D_x^{\alpha}f(u)(u'_x)^{\alpha}, \tag{4}$$

which are direct consequences of the equality $d^{\alpha}x(t) = \Gamma(1+\alpha)dx(t)$.

Next, let us consider the time fractional differential equation with independent variables $x = (x_1, x_2,...,x_m, t)$ and a dependent variable u,

$$F(u, D_t^{\alpha}u, u_{x_1}, u_{x_2}, u_{x_3}, D_t^{2\alpha}u, u_{x_1x_1}, u_{x_2x_2}, u_{x_3x_3},...) = 0. \tag{5}$$

Using the variable transformation:

Corresponding Author: A. Golmamadian, Faculty of Mathematical Sciences, University of Gonbad, Iran.

$$u(x_1, x_2, ..., x_m, t) = U(\xi),$$
$$\xi = x_1 + l_1 x_2 + ... + l_{m-1} x_m + \frac{\lambda t^\alpha}{\Gamma(1+\alpha)}\beta \tag{6}$$

where k, l and λ are constants to be determined later; the fractional differential equation (5) is reduced to a nonlinear ordinary differential equation:

$$H = (U(\xi), U'(\xi), U''(\xi), ...), \tag{7}$$

wehere $"'" = \frac{d}{d\xi}$.

We assume that Eq. (7) has a solution in the form:

$$u(\xi) = \sum_{i=0}^{n} a_i F^i(\xi), \tag{8}$$

where a_i (i= 1,2,...,n) are real constants to be determined later. $F(\xi)$ expresses the solution of the auxiliary ordinary differential equation:

$$F'(\xi) = b + F^2(\xi), \tag{9}$$

Eq. (9) admits the following solutions:

$$F(\xi) = \begin{cases} -\sqrt{-b}\tanh(\sqrt{-b}\xi), & b \prec 0 \\ -\sqrt{-b}\coth(\sqrt{-b}\xi), & b \prec 0 \end{cases}$$
$$F(\xi) = \begin{cases} \sqrt{b}\tan(\sqrt{b}\xi), & b \succ 0 \\ -\sqrt{b}\cot(\sqrt{b}\xi), & b \succ 0 \end{cases}$$
$$F(\xi) = -\frac{1}{\xi}, \qquad b = 0 \tag{10}$$

Integer n in (8) can be determined by considering homogeneous balance [7] between the nonlinear terms and the highest derivatives of $u(\xi)$ in Eq. (7).

Substituting (8) into (7) with (9), then the left hand side of Eq. (7) is converted into a polynomial in $F(\xi)$, equating each coefficient of the polynomial to zero yields a set of algebraic equations for a_i,k,c. Solving the algebraic equations obtained and substituting the results into (8), then we obtain the exact traveling wave solutions for Eq. (1).

Application to nonlinear fractional Sharma-Tasso-Olever equation

We first consider the nonlinear fractional Sharma-Tasso-Olever equation [22].

$$D_t^\alpha u + 3au_x^2 + 3au^2 u_x + 3auu_{xx} + au_{xxx} = 0, \quad t \succ 0, 0 \prec \alpha \le 1 \tag{11}$$

Subject to the initial condition;

$$u(x,0) = -\sqrt{2\beta_0}\tan\left(\frac{\sqrt{2\beta_0}}{2}x\right), \tag{12}$$

where a and β_0 are arbitrary constants, α is a parameter describing the order of the fractional time derivative. The function $u(x,t)$ is assumed to be a causal function of time. For our purpose, we introduce the following transformations:

$$u(x,y,t) = U(\xi), \qquad \xi = x - \frac{\lambda t^\alpha}{\Gamma(1+\alpha)} \tag{13}$$

where λ is constant.

Substituting (20) into (18), we can know that (18) is reduced into an ordinary differential equation:

$$-\lambda U' + 3a(U')^2 + 3aU^2 U' + 3aUU'' + aU''' = 0 \tag{14}$$

Integrating Eq. (21) with respect to ξ yields:

$$B - \lambda U + 3aUU' + aU^3 + aU'' = 0 \tag{15}$$

According to the direct algebraic method at first we obtain homogeneous balance so we have m = 1.

Hence ,

$$U = A_1 F + A_0, \tag{16}$$

Substituting (23) into (22) and setting all the coefficients of powers of F to be zero, we obtain a system of nonlinear algebraic equations and by solving it, we obtain:

$$A_1 = -1, -2 \tag{17}$$

$Case1$: *if* $A_1 = -1$

$$A_0 = \pm\sqrt{\frac{\lambda + ab}{3a}}, B = \frac{1}{3}(2\lambda + 8ab) \tag{18}$$

$Case2$: *if* $A_1 = -2$

$$A_0 = \pm\sqrt{\frac{\lambda + 4ab}{3a}}, B = \frac{1}{3}(2\lambda + 14ab) \tag{19}$$

By substituting (24) and (25) in Eq. (23), we obtain:

$$U_1 = -\left[-\sqrt{-b}\tanh(\sqrt{-b}ik(x - \frac{\lambda t^\alpha}{\Gamma(1+\alpha)}))\right] \pm \sqrt{\frac{\lambda + ab}{3a}},$$

where $b \prec 0$ and k is an arbitrary real constant.

And

$$U_2 = -\left[-\sqrt{-b}\coth(\sqrt{-b}ik(x - \frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)})) \right] \pm \sqrt{\frac{\lambda + ab}{3a}},$$

where $b \prec 0$ and k is an arbitrary real constant.

$$U_3 = -\left[\sqrt{b}\tan(\sqrt{b}ik(x - \frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)})) \right] \pm \sqrt{\frac{\lambda + ab}{3a}},$$

where $b \succ 0$ and k is an arbitrary real constant.

$$U_4 = -\left[-\sqrt{b}\cot(\sqrt{b}ik(x - \frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)})) \right] \pm \sqrt{\frac{\lambda + ab}{3a}},$$

where $b \succ 0$ and k is an arbitrary real constant.

$$U_5 = \frac{1}{x - \frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)}} \pm \sqrt{\frac{\lambda}{3a}},$$

For $b = 0$.

Using the conditions (24) and (26) in Eq. (23), we obtain:

$$U_1 = -2\left[-\sqrt{-b}\tanh(\sqrt{-b}ik(x - \frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)})) \right] \pm \sqrt{\frac{\lambda + 4ab}{3a}},$$

where $b \prec 0$ and k is an arbitrary real constant.

And

$$U_2 = -2\left[-\sqrt{-b}\coth(\sqrt{-b}ik(x - \frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)})) \right] \pm \sqrt{\frac{\lambda + 4ab}{3a}}$$

where $b \prec 0$ and k is an arbitrary real constant.

$$U_3 = -2\left[\sqrt{b}\tan(\sqrt{b}ik(x - \frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)})) \right] \pm \sqrt{\frac{\lambda + 4ab}{3a}},$$

where $b \succ 0$ and k is an arbitrary real constant.

$$U_4 = -2\left[-\sqrt{b}\cot(\sqrt{b}ik(x - \frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)})) \right] \pm \sqrt{\frac{\lambda + 4ab}{3a}},$$

where $b \succ 0$ and k is an arbitrary real constant.

$$U_5 = \frac{2}{x - \frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)}} \pm \sqrt{\frac{\lambda}{3a}},$$

For $b = 0$.

CONCLUSION

The direct algebraic method is applied successfully for solving the system of nonlinear fractional differential equations. The performance of this method is reliable and effective and gives more solutions. This method has more advantages: it is direct and concise. Thus, we deduce that the proposed method can be extended to solve many systems of nonlinear fractional partial differential equations.

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