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# Noetherian Topological Spaces and Cohen-Macaulay

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**Abstract:** In this paper, R is commutative ring with identity and L is K-lattice. We prove that, if p ⊆ Spec (L) and each A ∈ L is p-radical finite, then p is sequential Noetherian topological space and is s-compact. Furthermore, it has only a finite number of distinct irreducible components. Also, it is shown that if the lattice of ideals R is a principal lattice, then the prime spectrum Spec(R) is a sequential Noetherian topological space. Finally, it is shown that, if L(R) is a principal lattice and $\Im$ be the compact open of the Zariski spectrum of R, then R[$\Im$] is Cohen-Macaulay ring.

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## INTRODUCTION

The concept of Noetherian topological spaces arises naturally in the study of Noetherian rings and is of great interest in some areas of mathematics such as Algebraic Geometry. A topological space (X,τ) is called Noetherian if τ satisfies the ascending chain condition (ACC for short) : every strictly ascending chain $U_1 \subset U_2 \subset ...$ of elements of τ is finite [1-4]. Topological spaces that satisfy properties similar to ACC have been widely studied. In [5], spaces with Noetherian bases have been introduced (a topological space has a Noetherian base if it has a base that satisfies ACC) and many interesting results about such spaces have been obtained [5-7]. Clearly, every Noetherian space has a Noetherian base but the converse is not true in general. An element p in the K-lattice L is said to be prime if ab≤p implies a≤p or b≤p. Let Spec(L) denote the set of prime elements of L, which we give the Zariski topology.

If p⊆Spec(L), we always give p the relative topology induced from the Zariski topology on Spec(L).

If A∈L, we define the p-radical of A to be p-rad $(A) = \wedge \{\rho \in P | A \leq \rho\}$ and call A a p-radical element if A = P-rad(A). If p⊆Spec(L), we say that an element A of L is P-radically finite if there exists a compact element F≤A such that p-rad(F)=P-rad(A). In this paper, we study sequential properties of Noetherian topological spaces. The concept of s-compactness was introduced and studied by Gotchev. A topological space X is s-compact if every sequentially open cover of X has a finite subcover. In section 2, we present some needed results on multiplicative lattices with ACC on radical elements. A topological space X is irreducible if X is non-empty, and if any two non-empty open subsets of X intersect. In section 3, it shown that if p⊆Spec(L) and each A∈L is P-radically finite, then P is a sequential Noetherian topological space, is s- compact and it has only a finite number of distinct irreducible components. In section 4, it is pvoved that if L(R) is the lattice of ideals of a commutative ring R with identity, then an ideal A of R is principal as member of L(R) if and only if A is finitely generated and ARP is principal for each maximal ideal P of R. Also, we show that if L(R) is a principal lattice, then the prime spectrum Spec(R) is a sequential Noetherian topological space. Inparticular, it is shown that if L(R) is a principal lattice and $\Im$ be the compact open of the Zariski spectrum of R, then R[$\Im$] is Cohen-Macaulay.

**Ascending Chain Condition on Radical Elements:** We recall the de definitions. We follow the terminology of [8, 9]. Let (L, ≤) be a complete lattice with maximal element **R** and minimal element $0=0_L$. Then L is said to be multiplicative lattice if L is a multiplicative ordered monoid such that the multiplication on L distributes over arbitrary joins and such that **R** is the identity for the multiplication.

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**Definition 2.1:** **A** left lattice module over L, or simply an L-module, is a complete lattice M, together with a multiplication $L \times M \to M$ satisfying the following for $a,b \in L$, $\{a_\lambda | \lambda \in \Lambda\} \subseteq M$, $A \in M$, $\{B_\gamma | \gamma \in \Gamma\} \subseteq M$ :

(i) $(ab)A = a(bA)$;

(ii) $(\vee_\lambda a_\lambda)(\vee_\lambda B_\lambda) = \wedge_{\lambda,\gamma} a_\lambda B_\lambda$;

(iii) $\mathbf{R}A = A$;

(iv) $0_L A = 0_M$, where $0_M$ is the element smallest of M.

An element $A \in M$ is said to be compact if whenever $A \leq \vee\{B_\lambda | \lambda \in \Lambda\}$ for some family $\{B_\lambda | \lambda \in \Lambda\}$ of members of M, then there is a finite subset $\Gamma \subseteq \Lambda$ such that $A \leq \vee\{B_\lambda | \lambda \in \Gamma\}$. If each element of M the join of a family of compact elements of M, then M is said to be compactly generated or a CG-module. An L- module M is called a K-lattice if it is a CG-module and AH is compact for each compact element $A \in L$ and each compact element $H \in M$. We say that L is a CG-lattice or K-lattice respectively if this holds when L is considered as an L-module. Throughout this paper L will denote a K-lattice whose maximal element is compact and M will be an L-module which is also a K-lattice. In this section, will deal with various applications of lattice concepts to general topology-i.e., to the general theory of topological spaces. The ideas of general topology can be most simply introduced through the concept of a metric space. Through this paper L will denote a K - lattice whose maximal element is compact.

In a metric space M, a sequence $\{x_n\}$ of points is said to converge to the limit point a (in symbol,), if and only if $\lim_{n\to\infty} \delta(x_n,a) = 0$

In this section, we present some needed results on lattices with ACC on radical elements. See [10-12] for example, for results on rings with ACC on radical ideals. Let Spec(L) denote the set of prime elements of L, which we give the Zariski topology. That is, the closed sets are the sets of the from $V(A) = \{P \in \text{Spec}(L) | A \leq P \text{ with } A \in L\}$. If $p \subseteq \text{Spec}(L)$, we always give P the relative topology induced from the Zariski topology on Spec(L). Recall that a topological space P is said to be Noetherian if P satisfies the descending chain condition on closed sets. If $A \in L$, we define the P-radical of A to be $p\text{–}rad(A) = \wedge\{\rho \in p | A \leq \rho\}$ and call A a P-radical element if A = P-rad(A). If P = Spec(L), we will omit the P. It follows that a subset P of Spec(L) is Noetherian if and only if L has ACC on the P-radical elements of L. If $p \subseteq \text{Spec}(L)$, we say that an element A of L is P-radically finite if there exists a compact element $F \leq A$ such that P-rad(F) = P-rad(A). If P=Spec(L) we say radically finite for P-radically finite.

**Theorem 2.1:** If $p \subseteq \text{Spec}(L)$, then each $A \in L$ is P-radically finite if and only if P is a Noetherian topological space.

**Proof:** ($\Leftarrow$) Suppose there exists an $H \in L$ that is not P-radically finite. Let $h_1 \leq H$ be compact. Then $H \nleq \text{P-rad}(h_1)$. Let $h_2 \leq H$ be compact with $h_2 \nleq \text{rad}(h_1)$. Then $\text{Prad}(h_1) < \text{P-rad}(h_1 \vee h_2)$ and $H \nleq \text{P-rad}(h_1 \vee h_2)$, and so on, a contradiction to P being Noetherian.

($\Rightarrow$) To show that P is Noetherian, let $H_1 \leq H_2 \leq$...be a chain of P-radical elements. Let $H = \vee_{i=1}^{\infty} H_i$. Since H is P-radically finite, there exists a compact element $h \leq H$ such that P-rad(H)= P-rad(h). Then $h \leq H_j$ for some j and $\text{P-rad}(h) \leq H_j \leq \text{P-rad}(H) = \text{P-rad}(h)$. Therefore $H_j = H_k$ for all $k \geq j$.

**Irreducible Components and S-Compact on Radical Elements:** We begin this section with the definition of sequentially closed and sequentially open sets. A subset A of a topological space X is called sequentially closed if it has the following property: if a sequence in A converges in X to a point x, then $x \in A$. A subset E of a topological space X is called sequentially open if X\E is sequentially closed. For every $A \subset X$, we denote by $\overline{A}^s$ the sequential closure of A in X, which is the minimal sequentially closed set in X that contains A, and by $[A]_s$ the set A together with all limits of convergent sequences of points from the set A . A cover of a topological space X is called sequentially open if its elements are sequentially open sets. Recall, let $(X,\tau)$ be a topological space. The sequential topology $\tau_s$ is the topology on X such that a subset E of X is open in $(X,\tau)$ if and only if E is sequentially open in $(X,\tau)$. A topological space $(X,\tau)$ is called sequential if $\tau = \tau_s$. A topological space $(X,\tau)$ is called s-compact if $(X,\tau_s)$ is compact, or equivalently, every sequentially open cover of X has a finite subcover. A topological space X is irreducible if X is non-empty, and if any two non-empty open subsets of X intersect. Equivalently X is irreducible if $X \neq \emptyset$ and X is not the union of two closed subsets different from X. A subset Y of X is irreducible if it is an irreducible topological space with the induced topology. Let X be an irreducible topological space. If there is a point x in X such that $X = \overline{\{x\}}$ we call x a generic point of X, see ([13]).

If R is Noetherian ring then Spec R is a Noetherian topological space (see[2]) and every irreducible closed subset $F \subset \text{Spec}(R)$ has a unique generic point (see [3]).

Let X be a non-empty Noetherian topological space and $\Im(X)$ be the set of all irreducible closed subsets of X, ordered by inclusion. Let $\alpha$ be the supremum of all ordinals such that there exists a strictly increasing function $P:(0, \beta) \to \Im(X)$ We shall say that the hight of X is (denoted $h(X) = \alpha$) if $\alpha$ is an in finite ordinal and $h(X) = \alpha - 11$ otherwise. We define the hight of the empty set to be -1. It is clear that if $h(X)<\omega_0$ then $h(X) = \dim X$, where dim X is the dimension of the Noetherian topological space X defined to be the supremum of all integers n such that there exists a chain $Z_0 \subset Z_1 \subset \ldots \subset Z_n$ of distinct irreducible closed subsets of X (see [3]). Also, X is a Noetherian topological space in which every irreducible closed subset F has a generic point. Let X be a Noetherian topological space in which every irreducible closed subset F has a generic point. So, the space X is sequential if and only if $h(X) \leq \omega_1$.

**Theorem 3.1:** If $p \subseteq \mathrm{Spec}(L)$ and each $A \in L$ is P-radically finite, then P is sequential Noetherian topological space and is s-compact.

**Proof:** Let us suppose that $(X,\tau)$ is a Noetherian topological space but $(X,\tau_s)$ is not Noetherian. Then there exists a strictly decreasing by inclusion sequence $F_1 \supset F_2 \supset \ldots \supset F_n \supset \ldots$ of distinct sequentially closed subsets of X. For each we choose a point $x_n \in F_n/F_{n+1}$ and we form the sequence $(x_n)_n^\infty = 1$. Every sequence $(x_n)_n^\infty = 1$ in X has a convergent subsequence $(x_{n_k})_n^\infty = 1$ such that, the set of all $\lim_{k\to\infty} x_{n_k}$ is equal to the set $\overline{\{(x_{n_k})\}_n^\infty = 1}$ and the set $\overline{\{(x_{n_k})\}_n^\infty = 1}$ is irreducible. So, this sequence has a convergent subsequence $(x_{n_k})_n^\infty = 1$ such that the set of all its limit points $\lim_{k\to\infty} x_{n_k}$ is equal to the set $\overline{\{(x_{n_k})\}_n^\infty = 1}$. Then $x_{n_1} \in \lim_{k\to\infty} x_{n_k}$ and therefore $x_{n_1} \in \lim_{k\to\infty} x_{n_{k+1}}$. However, the set $\{x_{n_2}, x_{n_3}, \ldots, x_{n_{k+1}}, \ldots\}$ is a subset of the sequentially closed set $F_{n_2}$. Thus, $x_{n_1} \in \lim_{k\to\infty} x_{n_{k+1}} \subset F_{n_2}$ and hence $x_{n_1} \in F_{n_2}$. This is a contradiction because $x_{n_1} \in F_{n_1} \setminus F_{n_1+1} \subset F_{n_1} \setminus F_{n_2}$.

So, the prime spectrum Spec R is a sequential Noetherian topological space and every Noetherian topological space is s-compact. It follows immediately from Theorem 2.1.

**Definition 3.1:** The maximal irreducible subsets of X are called the irreducible components of X.

**Example:** The irreducible components of the topological space with the trivial topology is X itself. The irreducible components of the topological space X with the discrete topology are the points of X. The topological space X with the finite complement topology is irreducible exactly when X consists of infinitely many points, or consists of one point.

A Noetherian topological space X has only a finite number of distinct irreducible components $X_1, X_2, \ldots, X_n$. Moreover we have that X is not contained in $\cup_{i\neq j} X_j$ for $i=1,2,...,n$. Because, let I be the collection of all closed subsets of the topological space X. Assume that **I** is not empty. Since X is Noetherian the collection I then has a minimal element Y. Then Y can not be irreducible, so Y is the union $Y = Y' \cup Y''$ of two closed subsets Y' and Y'' different from Y. By the minimality of Y the sets Y' and Y'' both have a finite number of irreducible components. Consequently Y can be written as a union of a finite number of closed irreducible subsets.

So Y has only a finite number of irreducible components. This contradicts the assumption that **I** is not empty. Hence **I** is empty and the Proposition holds. If i is such that $X_i \subseteq \cup_{i\neq j} X_j$ we have that $X_i$ is covered by the closed subsets $X_i \cap X_j$ for $i \neq j$. Since $X_i$ is irreducible it follows that $X_i$ must be contained in one of the $X_j$, which contradicts the maximality of $X_i$.

**Corollary 3.1:** If $P \subseteq \mathrm{Spec}(L)$ and each $A \in L$ is P-radically finite, then P has only a finite number of distinct irreducible components $P_1, P_2, \ldots, P_n$.

**Cohen-Macaulay Ring and Noetherian Topological Space:** Dilworth overcame this in [14], with a new notion of a principal element. Basically, an element E of a multiplicative lattice L, is said to be meet-(join) principal if $(A \wedge (B:E))E = (AE) \wedge B$ (if $(BE \vee A):E = B \vee (A:E)$) for all A and B in L. A principal element is an element that is both meet-principal and join-principal or $A \wedge E = (A:E)E$ and $AE:E = A \vee (0:E)$, for all $A \in L$. A lattice L, is called a principal lattice, when each of its elements is principal. Here, the residual quotient of two elements A and B is denoted by A:B, so $A:B = \vee\{X \in L | XB \leq A\}$. The following theorem is proved in [15].

**Theorem 4.1:** Let R be a commutative ring with identity. Then L(R) is a principal lattice, if and only if, R is a Noetherian multiplication ring.

**Theorem 4.2:** Let R be a commutative ring with identity. If L(R) is a principal lattice, then the prime spectrum Spec(R) is a sequential Noetherian topological space.

**Proof:** It follows immediately from Theorem 4.1 and Theorem 3.1.

**Definition 4.1:** Let R be a commutative ring, and P a finite poset (= partially ordered set). We say that A is a ASL (algebra with straightening lows) on P over R if the followings hold.

**ASL-0:** An injective map P→A is given, A a graded R-algebra generated by P, and each element of P homogeneous of positive degree. We call a product of elements of P a monomial in P. Formally, a monomial M is a map P→$N_0$ and we denote $M = \prod_{x \in p} x^{M(x)}$ so that it also stands for an element of A. A monomial in P of the form

$x_{i_1} \ldots x_{i_l}$

with $x_{i_1} \le \ldots \le x_{i_l}$ is called standard.

**ASL-1:** The set of standard monomials in P is an R-free basis of A.

**ASL-2:** For x, y∈P such that x≰y and y≰x, there is an expression of the form

$$xy = \sum_M c_M^{xy} M \left( c_M^{xy} \in R \right)$$

where the sum is taken over all standard monomials M = $x_1, \ldots, x_{r_M}$ $(x_1 \le \ldots \le x_{r_M})$ with $x_1$<x, y and deg M = deg (xy).

The most simple example of an ASL on P over R is the Stanley-Reisner ring R[P] = R[x | x∈P] / (xy | x≰y, y≰x). The Stanley-Reisner rings play central role in theory of ASL.

**Theorem 4.3:** [16] If R is Cohen-Macaulay ring and P is a distributive lattice, then R[P] is Cohen-Macaulay.

**Theorem 4.4:** Let R be commutative ring with an identity. If L(R) is a principal lattice and $\mathfrak{I}$ be the compact open of the Zariski spectrum of R. Then R[$\mathfrak{I}$] is Cohen-Macaulay.

**Proof:** The set of all prime ideals of a ring R has a natural topology with basic open

D(a) = p{p|a ∉ p}

We clearly have

D(a)∩D(b) = D(ab), D(0) = $\phi$

The space of all topology, in general non Hausdorff. Though we cannot describe the points of this space effectively in general, we can describe the topology of the space effectively. The compact open of the spectrum are of the form

$$D(a_1, \ldots, a_n) = D(a_1) \cup \ldots \cup D(a_n)$$

The compact open form a distributive lattice. By Theorem 4.2, R[$\mathfrak{I}$] is Cohen-Macaulay.

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