Analytical Solution for Two-Dimensional Fractional Dispersion Equation by Modified Decomposition Method

I.I. Gorial
Department of Mathematics, Ibn Al–Haitham College Education, Baghdad University, Iraq

 Submitted: Jan 23, 2013; Accepted: Mar 5, 2013; Published: Jul 22, 2013

Abstract: In this paper, an analytical solution of the fractional dispersion equation has been presented. The algorithm for the analytical solution for this equation is based on modified decomposition method. The fractional derivative is described in Caputo's sense. The analytical method has been applied to solve a practical example and the results have been compared with exact solution.

Key words: Modified Decomposition method • Fractional derivative • Fractional dispersion equation

INTRODUCTION

The fractional calculus is used in many fields of science and engineering [1-5]. The solution of differential equation containing fractional derivatives is much involved and its classic a analytic methods are mainly integral transforms, such as Laplace transform, Fourier transform, Mellin transform, etc. [1, 2, 6].

In recent years, Adomian decomposition method is applied to solving fractional differential equations. This method efficiently works for initial value or boundary value problems, for linear or nonlinear, ordinary or partial differential equations and even for stochastic systems [7] as well. By using this method Saha Ray et al. [8-10, 13] solved linear differential equations containing fractional derivative of order 1/2 or 3/2 and nonlinear differential equation containing fractional derivative of order 1/2.

In this paper, we consider the two-dimensional fractional dispersion equation of the form: [14]

\[
\frac{\partial^\alpha u(x, y, t)}{\partial t^\alpha} = a(x, y) \frac{\partial^\alpha u(x, y, t)}{\partial x^\alpha} + b(x, y) \frac{\partial^\beta u(x, y, t)}{\partial y^\beta} + q(x, y, t)
\]

(1)

on a finite rectangular domain \(X_l < x < x_r\) and \(y_l < y < y_r\), with fractional orders \(1 < \alpha < 2\) and \(1 < \beta < 2\), where the diffusion coefficients \(a(x, y) > 0\) and \(b(x, y) > 0\). The ‘forcing’ function \(q(x, y, t)\) can be used to represent sources and sinks. We will assume that this fractional dispersion equation has a unique and sufficiently smooth solution under the following initial and boundary conditions. Assume the initial condition \(u(x, y, 0) = f(x, y)\) for \(X_l < x < x_r\) and \(y_l < y < y_r\) and Dirichlet boundary condition \(u(x, y, t) = k(x, y, t)\) on the boundary (perimeter) of the rectangular region \(X_l < x < X_R\) and \(y_l < y < y_R\), with the additional restriction that \(K(x, y, t) = K(x, y, t) > 0\.

Eq.(1) also uses Caputo fractional derivatives of order \(\alpha\) and \(\beta\).

In the present work, we apply the modified decomposition method for solving eq. (1) and compare the results with exact solution. The paper is organized as follows. In section 2, mathematical aspects. In section 3, basic idea of modified decomposition method. In section 4 the two dimensional fractional dispersion equation and its solution by modified decomposition method. In section 5 numerical example is solved using the modified decomposition method. Finally, we present conclusion about solution the two dimensional fractional dispersion equations in section 6.

Mathematical Aspects: The mathematical definition of fractional calculus has been the subject of several different approaches [15, 16]. The Caputo fractional derivative operator \(D^\alpha\) of order \(\alpha\) is defined in the following form:

\[
D^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \int_{x_0}^{x} \frac{f^{(m)}(t)}{(x-t)^{\alpha+m}} dt, \quad \alpha > 0,
\]

where \(m-1 < \alpha < m, m \in \mathbb{N}, x > 0\.

Similar to integer-order differentiation, Caputo fractional derivative operator is a linear operation.
\[ D^\alpha(\lambda f(x) + \mu g(x)) = \lambda D^\alpha f(x) + \mu D^\alpha g(x) \]

where \( \lambda \) and \( \mu \) are constants.

For the Caputo's derivative we have:

\[ D^\alpha (x-L)^+ = \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)}(x-L)^{\alpha-n} \]

and

\[ D^\alpha (R-x)^+ = \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)}(R-x)^{\alpha-n} \]

**Basic Idea of Modified Decomposition Method:** Consider a general nonlinear equation, \([17]\)

\[ Lu + R(u) + F(u) = g(t) \quad (2) \]

where \( L \) is the operator of the highest-ordered derivatives with respect to \( t \) and \( R \) is the remainder of the linear operator. The nonlinear term is represented by \( F(u) \). Thus we get

\[ Lu = g(t) - R(u) - F(u) \quad (3) \]

The inverse \( L^{-1} \) is assumed an integral operator given by

\[ L^{-1} = \int g(t) \, dt \]

The operating with the operator \( L^{-1} \) on both sides of Equation (3) we have

\[ u = f + L^{-1}(g(t) - R(u) - F(u)) \quad (4) \]

where \( f \) is the solution of homogeneous equation

\[ Lu = 0 \quad (5) \]

involving the constants of integration. The integration constants involved in the solution of homogeneous Equation (5) are to be determined by the initial or boundary condition according as the problem is initial-value problem or boundary-value problem.

The ADM assumes that the unknown function \( u(x, t) \) can be expressed by an infinite series of polynomials given by

\[ u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \]

and the nonlinear operator \( F(u) \) can be decomposed by an infinite series of polynomials given by

\[ F(u) = \sum_{n=0}^{\infty} A_n \]

where \( u_n(x, t) \) will be determined recurrently and \( A_n \) are the so-called polynomials of \( u_n, u_{n+1}, \ldots \)

\[ A_n = \frac{1}{n!} \frac{d^n}{dx^n} \left[ F\left( \sum_{\nu=0}^n \lambda^\nu u_\nu \right) \right] \quad n = 0, 1, 2, \ldots \]

But the modified decomposition method was introduced by Wazwaz \([18]\). This method is based on the assumption that the function \( f(x) \) can be divided into two parts, namely \( f_1(x) \) and \( f_2(x) \). Under this assumption we set

\[ f(x) = f_1(x) + f_2(x) \]

We apply this decomposition when the function \( f \) consists of several parts and can be decomposed into two different parts. In this case, \( f \) is usually a summation of a polynomial and trigonometric or transcendental functions. A proper choice for the part \( f_1 \) is important. For the method to be more efficient, we select \( f_1 \) as one term of \( f \) or at least a number of terms if possible and \( f_2 \) consists of the remaining terms of \( f \).

**Modified Decomposition Method for Solving Two-Dimensional Fractional Dispersion Equation:**

We adopt modified decomposition method for solving Equation (1). In the light of this method we assume that

\[ u = \sum_{n=0}^{\infty} u_n \]

Now, Equation (1) can be rewritten as

\[ Lu(x,t) = a(x,y) \frac{\partial^\alpha u(x,y,t)}{\partial x^\alpha} + b(x,y) \frac{\partial^\beta u(x,y,t)}{\partial y^\beta} + q(x,y,t) \]

where \( L \), which is an easily invertible linear operator, \( \frac{\partial^\alpha}{\partial x^\alpha} \) is the caputo derivative of order \( \alpha \).
Therefore, we can write,

\[ u(x,y,t) = u(x,y,0) + L_1^+ \left( a(x,y) \sum_{n=0}^{\infty} \frac{\partial^{\alpha} u_n}{\partial x^\alpha} \right) + L_1^+ \left( b(x,y) \sum_{n=0}^{\infty} \frac{\partial^{\beta} u_n}{\partial y^\beta} \right) + L^+(q(x,y,t)) \]

(6)

| x = y | t | Exact Solution | Decomposition Method | \(|u_n-u_{n+1}|_{MADM}| |
|-------|---|---------------|---------------------|---------------------|
| 0     | 1 | 0.00000000000 | 0.00000000000       | 0.00000000000 |
| 0.1   | 1 | 0.00000009241 | 0.00000009241       | 0.00000000000 |
| 0.2   | 1 | 0.00000089640 | 0.00000089640       | 0.00000000000 |
| 0.3   | 1 | 0.00001302000 | 0.00001302000       | 0.00000000000 |
| 0.4   | 1 | 0.00000869600 | 0.00000869600       | 0.00000000000 |
| 0.5   | 1 | 0.00379200000 | 0.00379200000       | 0.00000000000 |
| 0.6   | 1 | 0.01300000000 | 0.01300000000       | 0.00000000000 |
| 0.7   | 1 | 0.03500000000 | 0.03500000000       | 0.00000000000 |
| 0.8   | 1 | 0.08400000000 | 0.08400000000       | 0.00000000000 |
| 0.9   | 1 | 0.18400000000 | 0.18400000000       | 0.00000000000 |
| 1     | 1 | 0.36800000000 | 0.36800000000       | 0.00000000000 |
| 0     | 2 | 0.00000000000 | 0.00000000000       | 0.00000000000 |
| 0.1   | 2 | 0.00000003399 | 0.00000003399       | 0.00000000000 |
| 0.2   | 2 | 0.00000032980 | 0.00000032980       | 0.00000000000 |
| 0.3   | 2 | 0.00000479100 | 0.00000479100       | 0.00000000000 |
| 0.4   | 2 | 0.00031990000 | 0.00031990000       | 0.00000000000 |
| 0.5   | 2 | 0.00139500000 | 0.00139500000       | 0.00000000000 |
| 0.6   | 2 | 0.00461300000 | 0.00461300000       | 0.00000000000 |
| 0.7   | 2 | 0.01300000000 | 0.01300000000       | 0.00000000000 |
| 0.8   | 2 | 0.03100000000 | 0.03100000000       | 0.00000000000 |
| 0.9   | 2 | 0.06800000000 | 0.06800000000       | 0.00000000000 |
| 1     | 2 | 0.13500000000 | 0.13500000000       | 0.00000000000 |
| 0     | 3 | 0.00000000000 | 0.00000000000       | 0.00000000000 |
| 0.1   | 3 | 0.00000001251 | 0.00000001251       | 0.00000000000 |
| 0.2   | 3 | 0.00000012130 | 0.00000012130       | 0.00000000000 |
| 0.3   | 3 | 0.00000017620 | 0.00000017620       | 0.00000000000 |
| 0.4   | 3 | 0.00000177700 | 0.00000177700       | 0.00000000000 |
| 0.5   | 3 | 0.00051320000 | 0.00051320000       | 0.00000000000 |
| 0.6   | 3 | 0.00171000000 | 0.00171000000       | 0.00000000000 |
| 0.7   | 3 | 0.00472900000 | 0.00472900000       | 0.00000000000 |
| 0.8   | 3 | 0.01100000000 | 0.01100000000       | 0.00000000000 |
| 0.9   | 3 | 0.02500000000 | 0.02500000000       | 0.00000000000 |
| 1     | 3 | 0.05000000000 | 0.05000000000       | 0.00000000000 |

In [11], he assumed that if the zeroth component \( u_0 = f \) and the function \( f \) is possible to divide into two parts such as \( f_1 \) and \( f_2 \), then one can formulate the recursive algorithm for \( u_n \) and general term \( u_{n+1} \) in a form of the modified decomposition method recursive scheme as follows:

\[ u_n = f_1 + L^1 \left( a(x,y) \sum_{n=0}^{\infty} \frac{\partial^{\alpha} u_n}{\partial x^\alpha} \right) \]

\[ + L^1 \left( b(x,y) \sum_{n=0}^{\infty} \frac{\partial^{\beta} u_n}{\partial y^\beta} \right) + L^1 (q(x,y,t)) \]

Numerical Application: In this section, we apply modified decomposition method for finding the analytical solution of fractional dispersion equation:

\[ \frac{\partial u(x,y,t)}{\partial t} = a(x,y) \frac{\partial^{\alpha} u(x,y,t)}{\partial x^\alpha} + b(x,y) \frac{\partial^{\beta} u(x,y,t)}{\partial y^\beta} + q(x,y,t) \]

with the coefficient function: \( a(x,y) = \Gamma(2.2)x^{2.8}y/6 \) and \( b(x,y) = 2x^{2.8}y/\Gamma(4.6) \) and the source function: \( q(x,y,t) = (1 + 2xy)e^{x^{1.3}} \), subject to the initial condition \( u(x,y,0) = x^{1.3}y^{1.6} \), \( 0 < x < 1 \) and Dirichlet boundary conditions \( u(x,0,t) = u(0,y,t) = 0, \) \( u(x,1,t) = e^{-x} \) and \( u(1,y,t) = e^{-y} \). Note that the exact solution to this problem is: \( u(x,y,t) = e^{-x^{1.3}y^{1.6}} \).

Table 1 shows the analytical solutions for fractional dispersion equation obtained for different values and comparison between exact solution and analytical solution.

CONCLUSION

- Analytical solutions for fractional dispersion equation obtained for different values of a using modified decomposition method has been described and demonstrated.
- It is clear that the modified decomposition method is in high agreement with the exact solutions.
REFERENCES