World Applied Sciences Journal 22 (4): 532-537, 2013
ISSN 1818-4952

DOI: 10.5829/idosi.wasj.2013.22.04.532

Application of the Harmonic Balance Method on Nonlinear Equations

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Abstract: In this article, The Newton Harmonic Balance Method (NHBM) is applied to investigate frequency and response of the systems with periodic behavior. This method is combined by the Harmonic Balance and Newton's methods. Results showed that for nonlinear vibration of oscillatory systems, only first-order approximation frequency has adequate precious. Two classical cases are used to illustrate the applicable of NHBM and results compared by other analytical methods and time marching solution results. It is predicted that NHBM will be find application in engineering specially vibration equations.

Key words: Nonlinear vibration % Oscillatory system % Newton Harmonic Balance Method % Time marching solution

INTRODUCTION

Nonlinear oscillations problem are important issues in physical science, mechanical structures and other engineering researches. The vibration response, the stability [1, 2] and the frequencies are basic items in oscillatory systems. So, investigating about the influence of various parameters of these items may be important in the design steps.

Most of real systems are modeled by nonlinear differential equations. Obtaining exact solution for these nonlinear problems is difficult and time consuming for researchers, thus scientists are tried to find new approaches to overcome this difficulties. Recently, many authors used different analytical methods to solve nonlinear equations in mechanical systems. Some kind of these methods like Homotopy Perturbation Method (HPM), Homotopy Analysis Method (HAM) and Variational Iteration Method (VIM) are powerful methods and can used for almost all types of nonlinear equations [3-14]. Some other methods like Frequency Amplitude Formulation (FAF), Max-Min Approach (MMA), Energy Balance Method (EBM), Harmonic Balance Method (HBM) and Newton Harmonic Balance Method (NHBM) are introduced for nonlinear oscillatory systems [15-23].

The Newton Harmonic Balance Method is composed of both Newton's Method and Harmonic Balance Method. Wu *et al*. [21] introduced this method and applied it on two examples. Lai *et al*. [22] is analyzed first-, second- and third-orders analytical approximation for second-order differential equation with cubic quantic nonlinearities. Also, they compared the obtained frequency with the exact frequency.

In this paper, the NBHM is applied on two nonlinear oscillatory systems and first- and second-order approximations of this method are investigated. The results obtained by the NHBM are compared with time marching solution results. Also, the influence of the initial amplitudes is scrutinized on the system response and stability.

Analysis, Solution Procedure, Results and Discussion of Cases: The application of Newton Harmonic Balance Method (NHBM) in mechanical structures especially oscillation systems investigated on two nonlinear vibration problems.

Case 1: Consider the motion equation of special Duffing-harmonic oscillator as follows [18]:

$$\ddot{u}+\frac{u^3}{1+u^2}=0 \qquad \rightarrow \qquad \ddot{u}(1+u^2)+u^3=0 \tag{1}$$

Under the transformation $J = Tt$, the Eq. (1) can be written as:

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$$w^2 u''(t)(1+u^2(t)) + u^3(t) = 0 \tag{2}$$

where T is angular frequency and prime denotes differentiation with respect to J.

Also, initial condition is:

$$u(0) = A \qquad u'(0) = 0 \tag{3}$$

where A denotes the maximum amplitude.

With second order approximation, $u(J)$ and T^2 may be extend as follows [21, 22]:

$$u(J) = u_1(J) + \,)\, u_1(J) \tag{4}$$

$$w^2 = w_1^2 + \Delta w_1^2 \tag{5}$$

Substituting Eq. (4) and Eq. (5) into Eq. (3), we have:

$$(w_1^2 + \Delta w_1^2)(u_1'' + \Delta u_1'')(1 + (u_1 + \Delta u_1)^2) + (u_1 + \Delta u_1)^3 = 0 \tag{6}$$

According to initial conditions and for first order approximation, we set:

$$u_1(t) = A\cos t\,, \qquad \Delta u_1 = \Delta u'' = \Delta w_1^2 = 0 \tag{7}$$

With substitute Eq. (7) into Eq. (6) and avoiding the presence of secular terms, the angular frequency for first order approximation obtained and written as follows:

$$-w_1^2 + \frac{3}{4}A^2 - \frac{3}{4}A^2 w_1{}^2 = 0 \quad \Rightarrow \quad w_1 = \sqrt{\frac{3A^2}{4+3A^2}} \tag{8}$$

For the second analytical approximation, we set:

$$\Delta u_1 = c(\cos t - \cos 3t) \tag{9}$$

Substituting Eq. (9) into Eq. (6) and expanding the achieved expression in a trigonometric series and then putting the coefficients of cos3J and cos3J equal to zero, Eq. (10) and Eq. (11) obtained.

$$(-3A^3 + 4A)\Delta w_1^2 + c(-6A^2 + 4w_1^2 - 2A^2 w_1^2) + 4Aw_1^2 - 3A^3 + 3A^3 w_1^2 = 0 \tag{10}$$

$$A^3 \Delta w_1^2 + c(3A^2 - 36w_1^2 - 19A^2 w_1^2) + A^3 w_1^2 - A^3 = 0 \tag{11}$$

Solving Eq. (10) and Eq. (11) simultaneously, it is obtained:

$$w_1^2 = -\frac{-124A^2w_1^2 - 70A^4w_1^2 + 188A^2w_1^4 + 55A^4w_1^4 + 144w_1^4 + 15A^4}{-15A^4 + 188A^2w_1^2 + 55A^4w_1^2 - 12A^2 + 144w_1^2} \tag{12}$$

$$c = -\frac{4A^3}{-15A^4 + 188A^2w_1^2 + 55A^4w_1^2 - 12A^2 + 144w_1^2} \tag{13}$$

From Eq. (4) and Eq. (5) and using second order analytical solution the angular frequency and the system displacement may be written as:

$$w = \sqrt{\frac{3A^2}{4+3A^2} - \frac{-124A^2w_1^2 - 70A^4w_1^2 + 188A^2w_1^4 + 55A^4w_1^4 + 144w_1^4 + 15A^4}{-15A^4 + 188A^2w_1^2 + 55A^4w_1^2 - 12A^2 + 144w_1^2}} \tag{14}$$

$$u(t) = (A + c)\cos t - (c)\cos 3t \tag{15}$$

where c is evaluated from Eq. (13).

RESULT AND DISCUSSION

Motion equation of special Duffing-harmonic oscillator investigated and solved with NHBM. The frequency obtained by applying second order approximation of NHBM (Eq. (14)) compared with exact solution and other analytical solutions in Table 1. Also, time histories of system displacement for two different initial amplitude illustrated using NHBM and time marching solution in Fig. 1. From this figure and Table 1,

Table 1: Comparison between NHBM obtained frequency with frequencies obtained in other literatures.

A	He [24]	Mickens [25]	Tiwari *et al.* [26]	Turgut *et al.* [18]	Present study Eq. (14)	Exact solution [18]
0.1	0.0863	0.0844	0.0862	0.0862	0.0844	0.0844
1	0.6547	0.6464	0.6436	0.6516	0.6383	0.6368
10	0.9934	0.9931	0.9910	0.9931	0.9921	0.9909

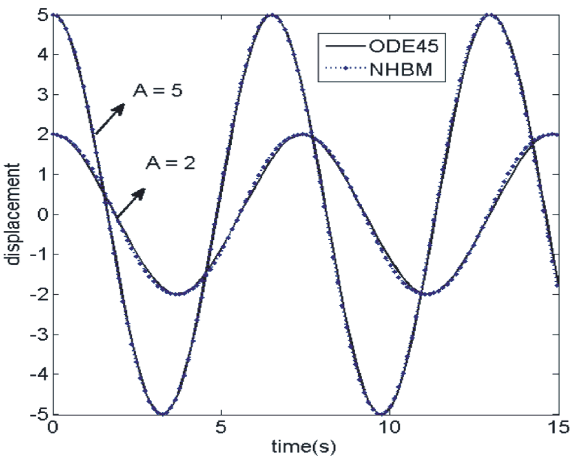

Fig. 1: System displacement for various A.

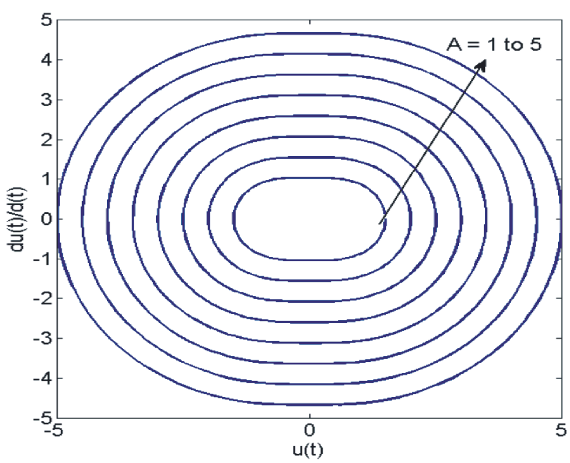

Fig. 2: Phase plan maps for showing the influence of A in the system stability.

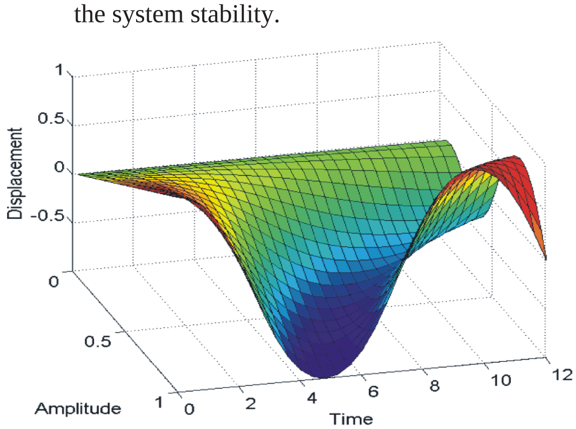

Fig. 3: Influence of initial amplitude on time histories of NHBM displacement.

NHBM has excellent agreement with other analytical or exact results and provides suitable approximating for this type of problems.

Fig. 2 shows the phase plane of the system which indicates displacement versus velocity and showing the system stability. In addition Fig. 3 shows the displacement behavior of the system versus time and initial amplitude.

Case 2: Consider the following nonlinear oscillator [20]:

$$\ddot{u} + \frac{1}{u} = 0, \qquad \rightarrow \qquad u^2\ddot{u} + u = 0 \tag{16}$$

By substitute $J = Tt$ Eq. (16) changes to:

$$w^2u^2(t)u''(t) + u(t) = 0 \tag{17}$$

where initial conditions are:

$$u(0) = A \qquad u'(0) = 0 \tag{18}$$

Here A is the maximum amplitude of the system.

Substituting Eq. (4) and Eq. (5) into Eq. (17), it is obtained:

$$(w_1^2 + \Delta w_1^2)(u_1 + \Delta u_1)^2(u_1'' + \Delta u_1'') + (u_1 + \Delta u_1) = 0 \tag{19}$$

Linearizing Eq. (19) respect to $)u_1$ and Δw_1^2 yield:

$$u_1''(u_1^2w_1^2 + 2u_1w_1^2\Delta u_1 + u_1^2\Delta w_1^2) + u_1^2w_1^2\Delta u_1'' + u_1 + \Delta u_1 = 0 \tag{20}$$

With substitute Eq. (7) into Eq. (20) for first order approximation and avoiding the presence of secular terms, the angular frequency may be written as:

$$-A^2w_1^2\cos^3 t + \cos t = 0 \quad \Rightarrow \quad w_1 = \sqrt{\frac{4}{3}}A^{-1} \tag{21}$$

For the second analytical approximation, by substituting Eq. (9) into Eq. (20) and expanding the obtained expression in a trigonometric series, then by putting the coefficients of cos$J$ and cos3J equal to zero, results achieved in a set of simultaneous equations in terms of Δw_1^2 and c:

$$-3A^3\Delta w_1^2 + c(4+2A^2w_1^2) + 4A - 3A^3w_1^2 = 0 \quad (22)$$

$$-A^3\Delta w_1^2 + c(-4+19A^2w_1^2) - A^3w_1^2 = 0 \quad (23)$$

Solving Eq. (10) and (11) simultaneously, it is obtained:

$$\Delta w_1^2 = -\frac{16-92A^2w_1^2+55A^4w_1^4}{A^2(-16+55A^2w_1^2)} \quad (24)$$

$$c = \frac{4A}{-16+55A^2w_1^2} \quad (25)$$

Second order analytical approximate frequency and system response may be written as:

$$w = \sqrt{w_1^2+\Delta w_1^2} = \sqrt{\frac{4}{3A^2} - \frac{16-92A^2w_1^2+55A^4w_1^4}{A^2(-16+55A^2w_1^2)}} \quad (26)$$

$$u(t) = (A+c)\cos wt - (c)\cos 3wt \quad (27)$$

RESULT AND DISCUSSION

As indicated in previous case, NHBM is a strong method for solving systems with periodic behavior. The nonlinear oscillator introduced in Eq. (16) is periodic. The frequency obtained by NHBM using first order approximation and second order approximation in Eq. (21) and (26), respectively. The comparison of these frequencies with the exact frequency ($w_{ex} = \sqrt{2p}/2A$) and the frequency obtained with HBM ($w_{HBM} = \sqrt{162}/10A$), which both reported by Bele´ndez *et al*. [20] show in Table 2. From this table, it can be finding that the error of the NHBM frequencies comparison with exact frequency is constant for various initial amplitudes. The error of first order approximation frequency is 7.87 percent and the error of second order approximation frequency is 2.66 percent. Thus by using the higher order approximation, the precious of the NHBM became better and converges to the exact results.

Fig. 4, show schematically the first and second order frequency obtained with NHBM with exact solution frequency. Fig. 5 shows the comparison of the system response with time marching solution results at two

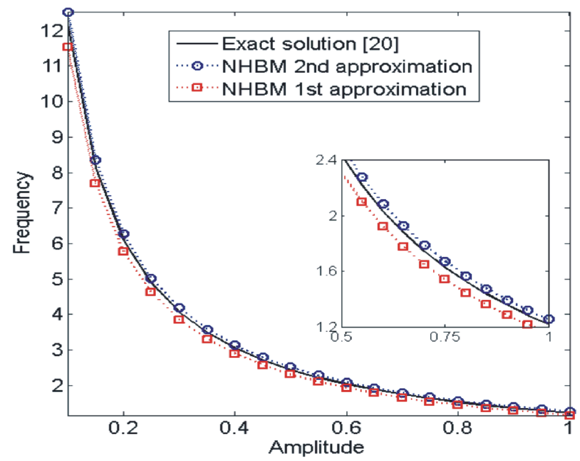

Fig. 4: Comparison of NHBM first and second order approximation with exact frequency.

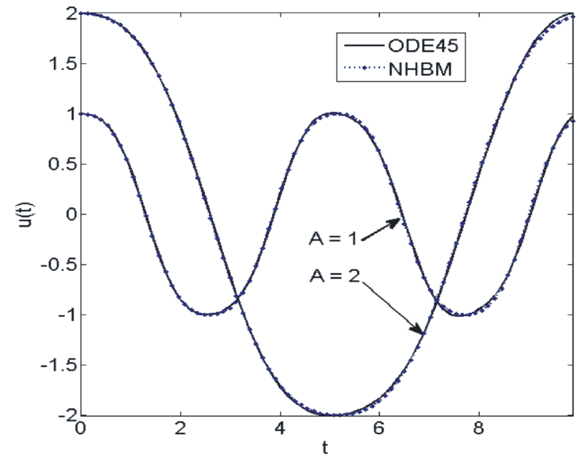

Fig. 5: Response of the system with NHBM and time marching solution for two different *A*.

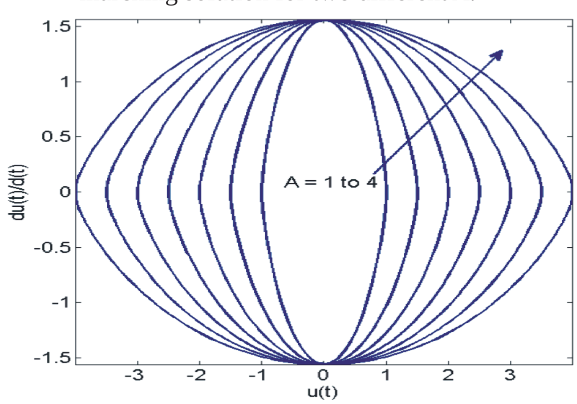

Fig. 6: Influence of *A* on the phase plane.

Table 2: Comparison of the frequency obtained via first and second order NHBM approximation with HBM and exact solution.

A	HBM [20]	Present study Eq. (21)	Present study Eq. (26)	Exact solution [20]
0.1	12.7279	11.5470	12.1999	12.5331
1	1.2728	1.1547	1.2200	1.2533
10	0.1273	0.1155	0.1220	0.1253

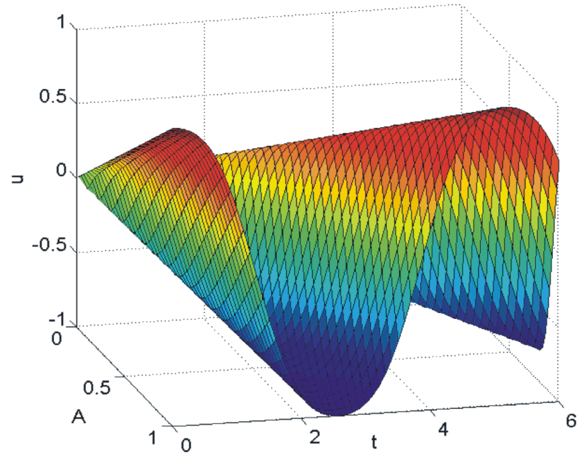

Fig. 7: Influence of initial amplitude on time histories of NHBM response.

different initial amplitudes. Also, influence of the initial amplitude on the phase plane of this case illustrated in Fig. 6. Furthermore, behavior of the system versus time and initial amplitude shows in Fig. 7.

CONCLUSION

In the present work, Newton Harmonic Balance Method applied to obtain analytical solution for nonlinear vibration in oscillatory systems. For this purpose, two problems with periodic behavior selected for investigating the effectiveness of this method. Results of the NHBM are compared with other analytical methods done in other literatures and time marching solutions. As indicated, the error of the studied systems is very worthless and the results confirmed the accuracy and the efficiently of the method. However, further research is needed to better understanding the effect of this method on engineering problems especially mechanical affairs.

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