World Applied Sciences Journal 22 (Special Issue of Applied Math): 23-29, 2013
ISSN 1818-4952

DOI: 10.5829/idosi.wasj.22.am.4.2013

Metric Aspects of $\mathcal{N}$-graphs

M. Akram[1], *Y.B. Jun*[2] *and F. Feng*[3]

1. Punjab University College of Information Technology, University of the Punjab, Lahore-54000, Pakistan
m.akram@pucit.edu.pk, makrammath@yahoo.com

2. Department of Mathematics Education (and RINS), Gyeongsang National University, Chinju 660-701
Republic of Korea
skywine@gmail.com

3. Department of Applied Mathematics, Xi'an University of Posts and Telecommunications, Xi'an, China
fengnix@hotmail.com

Abstract: We first define length, distance, radius, eccentricity, path cover and edge cover of an $\mathcal{N}$-graph. Then we introduce the concept of self centered $\mathcal{N}$-graphs and investigate some of their important properties. We also establish the necessary and sufficient conditions for a complete $\mathcal{N}$-graph to have an $\mathcal{N}$-bridge.

Key words: Self centered $\mathcal{N}$-graph, eccentricity, radius, diameter, path cover, edge cover, central vertex.

INTRODUCTION

In 1736, Euler first introduced the notion of graph theory. In the history of mathematics, the solution given by Euler of the well known K*ö*nigsberg bridge problem is considered to be the first theorem of graph theory. This has now become a subject generally regarded as a branch of combinatorics. The theory of graph is an extremely useful tool for solving combinatorial problems in different areas such as geometry, algebra, number theory, topology, operations research, optimization and computer science.

A (crisp) set A in a universe X can be defined in the form of its characteristic function $\mu_A : X \to \{0, 1\}$ yielding the value 1 for elements belonging to the set A and the value 0 for elements excluded from the set A. The most of the generalization of the crisp set have been introduced on the unit interval $[0, 1]$ and they are consistent with the asymmetry observation. In other words, the generalization of the crisp set to fuzzy sets relied on spreading positive information that fit the crisp point $\{1\}$ into the interval $[0, 1]$. Because no negative meaning of information is suggested, we now feel a need to deal with negative information. To do so, we also feel a need to supply mathematical tool. To attain such object, Jun *et al.* [1] have introduced a new function which is called negative-valued function (briefly, $\mathcal{N}$-function) to deal with negative information that fit the crisp point $\{-1\}$ into the interval $[-1, 0]$, and constructed $\mathcal{N}$-structures. It is important to be able to deal with negative information. It is noted that positive information represents what is granted to be possible, while negative information represents what is considered to be impossible. As an example, let us consider the spatial relations. Human beings consider "left" and "right" as opposite directions. But this does not mean that one of them is the negation of the other. The semantics of "opposite" captures a notion of symmetry rather than a strict complementation. In particular, there may be positions which are considered neither to the right nor to the left of some reference object.

In 1975, Rosenfeld [2] discussed the concept of fuzzy graphs whose basic idea was introduced by Kauffmann [3] in 1973. The fuzzy relations between fuzzy sets were also considered by Rosenfeld and he developed the structure of fuzzy graphs, obtaining analogs of several graph theoretical concepts. Bhattacharya [4] gave some remarks on fuzzy graphs. Akram et al. introduced the concepts of bipolar fuzzy graphs and interval-valued fuzzy line graphs [5-9]. In this paper, we first define length, distance, radius, eccentricity, path cover and edge cover of an $\mathcal{N}$-graph. Then we introduce the concept of self centered $\mathcal{N}$-graphs and investigate some of their important properties. We also establish the necessary and sufficient conditions for a complete $\mathcal{N}$-graph to have an $\mathcal{N}$-bridge. We have used standard definitions and terminologies in this paper. For other notations, terminologies and applications not mentioned in the paper, the readers are referred to [10-14].

PRELIMINARIES

A *graph* is an ordered pair $G^* = (V, E)$, where V is the set of vertices of G^* and E is the set of edges of G^*. Two vertices x and y in a graph G^* are said to be adjacent in G^* if $\{x, y\}$ is in an edge of G^*. (For simplicity an edge $\{x, y\}$ will be denoted by xy.) A *simple graph* is a graph without loops and multiple edges. A *complete graph* is a simple graph in which every pair of distinct vertices is connected by an edge. The complete graph on n vertices has n vertices and $n(n-1)/2$ edges. We will consider only graphs with the finite number of vertices and edges. An *isomorphism* of graphs G_1^* and G_2^* is a bijection between the vertex sets of G_1^* and G_2^* such that any two vertices v_1 and v_2 of G_1^* are adjacent in G_1^* if and only if $f(v_1)$ and $f(v_2)$ are adjacent in G_2^*. Isomorphic graphs are denoted by $G_1^* \simeq G_2^*$.

A *path* in a graph G is a sequence of vertices such that from each of its vertices there is an edge to the next vertex in the sequence. The *length* of a path $P : v_1 v_2 \cdots v_{n+1}$ $(n > 0)$ in G is n. A path $P : v_1 v_2 \cdots v_{n+1}$ in G is called a *cycle* if $v_1 = v_{n+1}$ and $n \geq 3$. An undirected graph G is *connected* if there is a path between each pair of distinct vertices. For a pair of vertices u, v in a connected graph G, the *distance* $d(u, v)$ between u and v is the length of a shortest path connecting u and v. The *eccentricity* $e(v)$ of a vertex v in a graph G is the distance from v to a vertex furthest from v, that is, $e(v) = \max\{d(u, v) \mid u \in V\}$. The *radius* of a connected graph (or weighted graph)G is defined as $\mathrm{rad}(G) = \min\{e(v) \mid v \in V\}$. The *diameter* of a connected graph (or weighted graph)G is defined as $\mathrm{diam}(G) = \max\{e(v) \mid v \in V\}$. The *eccentric set* S of a graph is its set of eccentricities. The *center* $C(G)$ of a graph G is the set of vertices with minimum eccentricity. A graph is *self-centered* if all its vertices lie in the center. Thus, the eccentric set of a self-centered graph contains only one element, that is, all the vertices have the same eccentricity. Equivalently, a self-centered graph is a graph whose diameter equals its radius.

Proposition 1. *Let G be a self-centered graph with n vertices, e edges, and diameter d.*

(1) *If $d = 1$, then $e = C(n, 2)$*

(2) *If d = 2 and n = 4, then e = 4*

(3) *If $d \geq 2$ and $n \geq 2d \neq 4$, then*

$$\ulcorner \frac{(nd - 2d - l)}{(d-1)} \urcorner \leq e$$

$$\leq \frac{(n^2 - 4nd + 5n - 4d^2 - 6d)}{2}.$$

If G is a self-centered graph with n vertices, e edges, and diameter 2, then $e \geq 2n - 5$.

Denote by $\mathcal{F}(X, [-1, 0])$ the collection of functions from a nonempty set X to $[-1, 0]$. We say that an element of $\mathcal{F}(X, [-1, 0])$ is a *negative-valued function* from X to $[-1, 0]$ (briefly, $\mathcal{N}$*-function* on X). By an $\mathcal{N}$*-structure* we mean an ordered pair (X, μ_1) of X and an $\mathcal{N}$-function μ_1 on X. By an $\mathcal{N}$*-relation* on X we mean an $\mathcal{N}$-function μ_2 on $X \times X$ satisfying the following inequality:

$$(\forall x, y \in X)(\mu_2(x, y) \geq \max\{\mu_1(x), \mu_1(y)\}), \quad (1)$$

where $\mu_1 \in \mathcal{F}(X, [-1, 0])$.

SELF CENTERED $\mathcal{N}$-GRAPHS

Definition 2. *By an $\mathcal{N}$-graph $G = < V, E, \mu_1, \mu_2 >$ of a graph $G^* = (V, E)$, we mean a pair $G = (\mu_1, \mu_2)$ where μ_1 is an $\mathcal{N}$-function in V and μ_2 is an $\mathcal{N}$-function on $E \subseteq V \times V$ such that*

$$\mu_{2ij} = \mu_2(\{x, y\}) \geq \max(\mu_1(x), \mu_1(y))$$

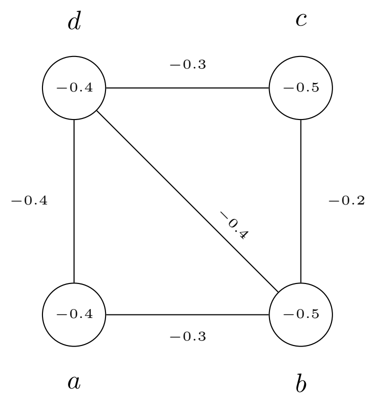
for all $\{x, y\} \in E$.

Throughout this paper, G^* is a crisp graph and G is an $\mathcal{N}$-graph.

Definition 3. *A path P in an $\mathcal{N}$-graph G is a sequence of distinct vertices $v_1, v_2, ..., v_n$ such that either one of the following conditions is satisfied:*
(1) $\mu_{2ij} = 0$ for some i, j.
(2) $\mu_{2ij} < 0$ for some i, j .

Definition 4. *An $\mathcal{N}$-graph G is connected if any two vertices are joined by a path. The μ- strength of a path $P : v_1 v_2 \cdots v_n$ is defined as $\max(\mu_2(v_i, v_j))$ for all i, j and is denoted by S_μ. If the edge possesses μ-strength value, then it is the strength of a path P. In other words, the strength of a path is defined to be the weight of the strongest edge of the path. That is the strength of a path is μ_{2ij} = S_μ.*

Example 5. *Consider an $\mathcal{N}$-graph G such that $V = \{a, b, c, d\}$, $E = \{(a, b), (a, d), (b, d), (b, c), (c, d)\}$.*



G

By routine computations, it is easy to see that:

- *ad is a path of length 1 and the strength is -0.4,*
- *abd is a path of length 2 and the strength is -0.3,*
- *$abcd$ is a path of length 3 and the strength is -0.2.*
- *A strongest path joining a and d is the path $P : bcd$.*

Definition 6. *Let G be a connected $\mathcal{N}$-graph. The $\mu-$length of a path $P : v_1 v_2 \cdots v_n$ in G, $l_\mu(P)$, is defined as $l_\mu(P) = \sum_{i=1}^{n-1} \frac{1}{\mu_2(v_i, v_{i+1})}$.*

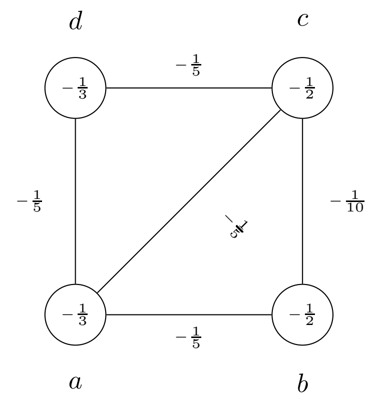
Definition 7. *Let G be a connected $\mathcal{N}$-graph. The μ-distance, $\delta_\mu(v_i, v_j)$, is the largest μ-length of any $v_i - v_j$ path P in G, where $v_i, v_j \in V$. That is, $\delta_\mu(v_i, v_j) = \max(l_\mu(P))$.*

Definition 8. *Let G be a connected $\mathcal{N}$-graph. For each $v_i \in V$, the μ-eccentricity of v_i, denoted by $e_\mu(v_i)$ and is defined as $e_\mu(v_i) = \max\{\delta_\mu(v_i, v_j) : v_i \in V, v_i \neq v_j\}$.*

Definition 9. *Let G be a connected $\mathcal{N}$-graph. The μ-radius of G is denoted by $r_\mu(G)$ and is defined as $r_\mu(G) = \max\{e_\mu(v_i) : v_i \in V\}$.*

Definition 10. *Let G be a connected $\mathcal{N}$-graph. The μ-diameter of G is denoted by $d_\mu(G)$ and is defined as $d_\mu(G) = \min\{e_\mu(v_i) : v_i \in V\}$.*

Example 11. *Consider an $\mathcal{N}$-graph G such that $V = \{a, b, c, d\}$, $E = \{(a, b), (a, c), (a, d), (b, c), (c, d)\}$.*



G

By routine computations, it is easy to see that:

(1) *ad is a path of length 1 and $l_\mu = -5$, acd is a path of length 2 and $l_\mu = -10$, $abcd$ is a path of length 3 and $l_\mu = -20$.*

(2)

$$\delta_\mu(a, d) = -5,\ \delta_\mu(a, b) = -5,\ \delta_\mu(a, c) = -5,$$

$$\delta_\mu(b, c) = -10,\ \delta_\mu(b, d) = -10,\ \delta_\mu(c, d) = -5.$$

(3) *μ-eccentricity of each vertex is*

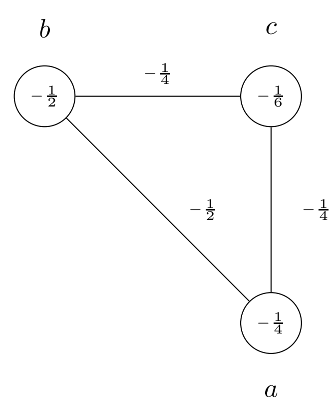
$$e_\mu(a) = -5,\ e_\mu(b) = -5,\ e_\mu(c) = -5,\ e_\mu(d) = -5.$$

(4) *Radius of G is -5, diameter of G is -5.*

Definition 12. *A vertex $v_i \in V$ is called a central vertex of a connected $\mathcal{N}$-graph G, if $r_\mu(G) = e_\mu(v_i)$ and the set of all central vertices of an $\mathcal{N}$-graph is denoted by $C(G)$.*

Definition 13. *A connected $\mathcal{N}$-graph G is a self centered graph, if every vertex of G is a central vertex, that is $r_\mu(G) = e_\mu(v_i), \forall v_i \in V$.*

Example 14. *Consider a bipolar graph G such that $V = \{a, b, c\}$, $E = \{ab, bc, ca\}$.*



Self centered $\mathcal{N}$-graph

By routine computations, it is easy to see that:

(i) *Distance is*

$$\delta_\mu(a,b) = -2,\ \delta_\mu(a,c) = -4,\ \delta_\mu(b,c) = -2.$$

(ii) *Eccentricity of each vertex is -4.*

(iii) *Radius of G is -4. Hence Hence G is self centered $\mathcal{N}$-graph.*

Definition 15. *A path cover of an $\mathcal{N}$-graph G is a set P of paths such that every vertex of G is incident to some path of P.*

Definition 16. *An edge cover of an $\mathcal{N}$-graph G is a set L of edges such that every vertex of G is incident to some edge of L. An $\mathcal{N}$-graph $H = (\mu_1', \mu_2')$ is said to be a $\mathcal{N}$-subgraph of a connected $\mathcal{N}$-graph G =, $(\mu)_1'(v_i) = \mu_1(v_i)$, $\forall v_i \in V'$ and $(\mu)_2'(v_i, v_j) = \mu_2(v_i, v_j), \forall(v_i, v_j) \in E'$.*

Definition 17. *$< C(G) >= H$ is an $\mathcal{N}$-subgraph of G induced by the central vertices of G, is called the center of G. A maximal connected subgraph of an $\mathcal{N}$-graph G is a subgraph that is connected and is not contained in any other connected subgraph of G. The components of an $\mathcal{N}$-graph G is its maximal connected subgraphs, where G is a disconnected $\mathcal{N}$-graph.*

Definition 18. *An $\mathcal{N}$-graph H is said to be an $\mathcal{N}$-subgraph of G induced by E' if $A' \subseteq A$ and $(\mu)_1'(v_i) = \mu_1(v_i)$, $\forall v_i \in V'$, $(\mu)_2'(v_i, v_j) = \mu_2(v_i, v_j)$, $\forall v_i, v_j \in V'$.*

Definition 19. *An $\mathcal{N}$-graph G is said to be a bipartite if the vertex set V can be partitioned into two non empty sets V_1 and V_2 such that*
(i) $\mu_2(v_i, v_j) = 0$, if $v_i, v_j \in V_1$ or $v_i, v_j \in V_2$.
(ii) $\mu_2(v_i, v_j) < 0$, if $v_i \in V_1$ or $v_j \in V_2$, for some i and j.

Definition 20. *A bipartite $\mathcal{N}$-graph G is said to be complete if $\mu_2(v_i, v_j) = \max(\mu_1(v_i), \mu_1(v_j))$, for all $v_i \in V_1$ and $v_j \in V_2$. It is denoted by $K_{V_{1i}, V_{2i}}$.*

Theorem 21. *In an $\mathcal{N}$-graph G for which $\mu_2 : V \times V \to [-1, 0]$ is not constant mapping, an edge (v_i, v_j) for which μ_{2ij} is minimum. Therefore it is a bridge of G.*

Theorem 22. *If G is an $\mathcal{N}$-bipartite graph then it has no strong cycle of odd length.*

Proof. Let G be an $\mathcal{N}$-bipartite graph with $\mathcal{N}$-bipartition V_1 and V_2. Suppose that it contains a strong cycle of odd length say $v_1, v_2, \cdots, v_n, v_1$ for some odd n. Without loss of generality, let $v_1 \in V_1$. Since (v_i, v_{i+1}) is strong for$i = 1, 2, \cdots, n-1$ and the nodes are alternatively in V_1 and V_2, we have v_n and $v_1 \in V_1$. But this implies that (v_n, v_1) is an edge in V_1, which contradicts the assumption that G is an $\mathcal{N}$-bipartite. Hence $\mathcal{N}$-bipartite graph has no strong cycle of odd length. □

Theorem 23. *Every complete $\mathcal{N}$-graph G is a self centered $\mathcal{N}$-graph and $r_\mu(G) = \frac{1}{\mu_{1i}}$, where μ_{1i} is the greatest.*

Proof. Let G be a complete $\mathcal{N}$-graph. To prove that G is self centered $\mathcal{N}$-graph. That is we have to show that every vertex is a central vertex. We claim that G is a μ-self centered $\mathcal{N}$-graph and $r_\mu(G) = \frac{1}{\mu_{1i}}$, where μ_{1i} is the greatest. Now choose some vertex $v_i \in V$ such that μ_{1i} is the greatest vertex membership value of G .
Case 1: consider all the $v_i - v_j$ path $P$of length n in G $\forall v_j \in V$.
(i) If $n = 1$, then $\mu_{2ij} = \max(\mu_{1i}, \mu_{1j}) = \mu_{1i}$. Therefore, the μ-length of P = $l_\mu(P) = \frac{1}{\mu_{1i}}$.
(ii) If $n > 1$, then one of the edges of P possesses the μ-strength μ_{1i} and hence $\mu-$ length of P will exceed $\frac{1}{\mu_{1i}}$. That is, $\mu-$ length of P = $l_\mu(P) > 1/\mu_{1i}$. Hence

$$\delta_\mu(v_i, v_j) = \min(l_\mu(P)) = \frac{1}{\mu_{1i}} \forall\ v_j \in V. \quad (2)$$

Case 2: Let $v_k \neq v_i \in V$. Consider all $v_k - v_j$ paths Q of length n in G, $\forall v_j \in V$.
(i)If $n = 1$, then $\mu_2(v_k, v_j) = \max(\mu_{1k}, \mu_{1j}) \leq \mu_{1i}$, since μ_{1i} is the greatest. Therefore μ-length of Q = $l_\mu(Q)$ = $\frac{1}{\mu_2(v_k,v_j)} \geq \frac{1}{\mu_{1i}}$.
(ii) If $n = 2$, $l_\mu(Q) = \frac{1}{\mu_2(v_k,v_{k+1})}$ + $\frac{1}{\mu_2(v_{k+1},v_j)} \geq \frac{2}{\mu_{1i}}$, since μ_{1i} is the greatest.
(iii) If $n > 2$, then $l_\mu(Q) \geq \frac{n}{\mu_{1i}}$ since μ_{1i} is the greatest. Hence,

$$\delta_\mu(v_k, v_j) = \min(l_\mu(Q)) \geq 1/\mu_{1i}\ \forall v_k, v_j \in V. \quad (3)$$

From Equation (2) and (3), we have,

$$e_\mu(v_i) = \min(\delta_\mu(v_i, v_j)) = \frac{1}{\mu_{1i}} \forall v_i \in V \quad (4)$$

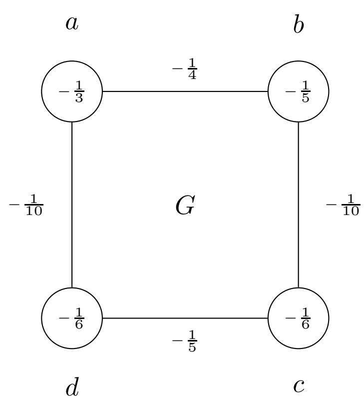
Hence G is a μ self centered $\mathcal{N}$-graph.

$$\begin{aligned} Now\ r_\mu(G) &= \min(e_\mu(v_i)) \\ &= \frac{1}{\mu_{1i}},\ \text{since by equation(6)} \\ r_\mu(G) &= \frac{1}{\mu_{1i}},\ \text{where } \mu_1(v_i) \text{ is greatest} \end{aligned}$$

From equation(2) and (4), every vertex of G is a central vertex. Hence G is a self -centered $\mathcal{N}$-graph. □

The following example shows that the converse of the above theorem is not true, in general.

Example 24. *Consider an $\mathcal{N}$-graph G such that $V = \{a, b, c, d\}$, $E = \{ab, ad, bc, cd\}$.*



Routine computations show that G is self centered $\mathcal{N}$-graph but not complete.

Lemma 25. *An $\mathcal{N}$-graph G is a self centered $\mathcal{N}$-graph if and only if $r_\mu(G) = d_\mu(G)$.*

Theorem 26. *If G is a complete $\mathcal{N}$-graph then for at least one edge $(\mu_2)^\infty(v_i, v_j)$ = $\mu_2(v_i, v_j)$.*

Proof. If G be a complete $\mathcal{N}$-graph. Consider a vertex v_i whose value is μ_{1i}.
Case (i): Let μ_{1i} be the greatest in the vertex $v_i \in V$. Let $v_i, v_j \in V$, then μ_{2ij} = μ_{1i} and $((\mu_{2ij}))^\infty$ = μ_{1i}. The strength of all the edges which are incident on the vertex v_i is μ_{1i}. Since G is a complete $\mathcal{N}$-graph.
Case (ii): Let μ_{1k} be the greatest, where $v_i \neq v_k$. Then $\mu_{2ik})$ = μ_{1k}. Since it is a complete $\mathcal{N}$-graph, there will be an edge between v_i and v_k, therefore, $(\mu_{2ik})^\infty = \mu_{1k}$. □

Theorem 27. *Let G be a complete $\mathcal{N}$-graph with n vertices. Then G has an $\mathcal{N}$-bridge if and only if there exist a decreasing sequence $\{s_1, s_2, ..., s_{n-1}, s_n\}$ such that $s_{n-2} > s_{n-1} \geq s_n$ where s_i = $\mu_1(v_i)$ $\forall$ i = 1,2,...,n. Also the arc (v_{n-1}, v_n) is an $\mathcal{N}$-bridge of G.*

Proof. Assume that G is a complete $\mathcal{N}$-graph and that G has an $\mathcal{N}$-bridge(v_i, v_j). Then we claim that there exist a decreasing sequence $\{s_1, s_2, ..., s_{n-1}, s_n\}$ such that $s_{n-2} > s_{n-1} \geq s_n$ where s_i = $\mu_1(v_i)$,$\forall$ $i = 1, 2, ..., n$.
Without loss of generality we assume that $\mu_1(v_i) \geq \mu_1(v_j)$, that is, $s_{n-1} \geq s_n$, where, $s_{n-1} = \mu_1(v_i)$, $s_n = \mu_1(v_j)$, so that $\mu_2(v_i, v_j)$ = $\mu_1(v_i)$. Assume to the contrary, that there is at least one vertex $v_i \neq v_k$ such that $\mu_1(v_i) \geq \mu_1(v_k)$, $s_{n-1} \geq s_k$, where $k \neq n$, $s_{n-1} = \mu_1(v_i)$, $s_k = \mu_1(v_k)$. Now consider the cycle C:$v_i v_j v_k v_i$ in G, then $\mu_2(v_i, v_j)$ =$\mu_2(v_i, v_k)$ = $\mu_1(v_i)$.

$$\mu_2(v_j, v_k) = \begin{cases} \mu_{1j} & \text{if } \mu_{1i} = \mu_{1j} \text{ or } \mu_{1i} > \mu_{1j} \geq \mu_{1k} \\ \mu_{1k} & \text{if } \mu_{1i} > \mu_{1k} > \mu_{1j} \end{cases}$$

In either case it is seen that μ_{1i} is the greatest among all other edges. But by Theorem 3.31, it is a contradiction to the fact that (v_i, v_j) is an $\mathcal{N}$-bridge.
Conversely, let $s_1 > s_2 > ... > s_{n-2} > s_{n-1} \geq s_n$ and $s_i = \mu_1(v_i)$.
Claim: (v_{n-1}, v_n) is an $\mathcal{N}$- bridge of G.
Now $\mu_2(v_{n-1}, v_n) = \max(\mu_1(v_{n-1}), \mu_1(v_n)) = \mu_1(v_{n-1})$ and clearly by hypothesis, all other edges of G will have μ- strength strictly greater than $\mu_1(v_{n-1})$. Hence the edge (v_{n-1}, v_n) is an $\mathcal{N}$-bridge by Theorem 3.20. □

Theorem 28. *Let G be a connected $\mathcal{N}$-graph with a path covers P of G. Then the necessary and sufficient condition for an $\mathcal{N}$-graph to be self centered $\mathcal{N}$-graph is*

$$\delta_\mu(v_i, v_j) = r_\mu(G), \forall\ (v_i, v_j)\ \in P. \tag{5}$$

Proof. We now assume that G is a self centered $\mathcal{N}$-graph and we have to prove that equation (5) holds. Suppose equation (5) does not hold, then we have, $\delta_\mu(v_i, v_j) \neq r_\mu(G)$ for some $(v_i, v_j) \in P$. By using Lemma 3.24, the above inequality becomes $\delta_\mu(v_i, v_j) \neq r_\mu(G)$ for some $(v_i, v_j) \in P$. Then $e_\mu(v_i) \neq r_\mu(G)$ for some $v_i \in V$ which implies G is not self centered $\mathcal{N}$-graph, which is a contradiction. Hence $\delta_\mu(v_i, v_j) = r_\mu(G)$ $\forall\ (v_i, v_j) \in P$.
Conversely, we now assume that equation (7) holds and we have to prove that G is a self centered $\mathcal{N}$-graph. If equation (7) holds, then we have $e_\mu(v_i) = \delta_\mu(v_i, v_j)$ $\forall\ (v_i, v_j) \in P$, which implies $e_\mu(v_i) = r_\mu(G)$ $\forall\ v_i \in V$. Hence G is self centered $\mathcal{N}$-graph. □

Corollary 29. *If G is a connected complete $\mathcal{N}$-graph with an edge cover L of G. Then the necessary and sufficient condition for an $\mathcal{N}$-graph to be self centered $\mathcal{N}$-graph is*

$$\delta_\mu(v_i, v_j) = r_\mu(G), \forall\ (v_i, v_j)\ \in L_2. \tag{6}$$

Theorem 30. *Let H be a connected μ-self centered $\mathcal{N}$-graph. Then there exist a connected $\mathcal{N}$-graph G such that $< C(G) >$ is isomorphic to H. Also $d_\mu(G) = 2r_\mu(G)$.*

Proof. Given that H be a connected μ-self centered $\mathcal{N}$-graph. Let $d_\mu(H) = m$. Then construct G from H as follows:

Take two vertices $v_i, v_j \in V$ with $\mu_1(v_i) = \mu_1(v_j) = \frac{1}{2m}$ and join all the vertices of H to both v_i and v_j with $\mu_2(v_i, v_k) = \mu_2(v_j, v_k) = \frac{1}{2m}$, $\forall\ v_k \in V'$. Put $\mu_1 = (\mu)'_1$, for all vertices in H and $\mu_2 = (\mu_2)'$ for all edges in H.
Claim: G is an $\mathcal{N}$-graph. Note that $\mu_1(v_i) \leq \mu_1(v_k)$, $\mu_1(v_j) \leq \mu_1(v_k)$, $\forall v_k \in V'$, since $d_\mu(H) = m$. Therefore $\mu_2(v_i, v_k) \leq \max(\mu_{1i}, \mu_{1k}) = \frac{1}{2m}$, similarly, $\mu_2(v_j, v_k) \leq \max(\mu_{1j}, \mu_{1k}) = \frac{1}{2m}$. Hence G is an $\mathcal{N}$-graph. Also, $e_\mu(v_k) = m\ \forall v_k \in V'$ and $e_\mu(v_i) = e_\mu(v_j) = \frac{1}{\mu_2(v_l, v_k)} = 2m\ \forall v_k \in V'$. Therefore, $r_\mu(G) = m,\ d_\mu(G) = 2m$. Hence $< C(G) >$ is isomorphic to H. □

Theorem 31. *An $\mathcal{N}$-graph G is a self centered if and only if $\delta_\mu(v_i, v_j) \geq r_\mu(G)$, $\forall v_i, v_j \in V$.*

Proof. We assume that G is self-centered $\mathcal{N}$-graph. That is, $e_\mu(v_i) = e_\mu(v_j)$, $\forall v_i, v_j \in V$, $r_\mu(G) = e_\mu(v_i)$, $\forall v_i \in V$. Now we want to show that $\delta_\mu(v_i, v_j) \geq r_\mu(G)$, $\forall v_i, v_j \in V$. By the definition of eccentricity, we obtain, $\delta_\mu(v_i, v_j) \geq e_\mu(v_i)$, $\forall v_i, v_j \in V$. This is possible only when $e_\mu(v_i) = e_\mu(v_j)$, $\forall v_i, v_j \in V$. Since G is self centered $\mathcal{N}$-graph, the above inequality becomes $\delta_{\mu^-}(v_i, v_j) \geq r_{\mu^-}(G)$.
Conversely, we now assume that $\delta_{\mu^-}(v_i, v_j) \geq r_{\mu^-}(G)$, $\forall v_i, v_j \in V$. Then we have to prove that G is self centered $\mathcal{N}$-graph. Suppose that G is not self centered $\mathcal{N}$-graph. Then $e_{\mu^-}(v_i) \neq r_{\mu^-}(G)$, for some $v_i \in V$. Let us assume that $e_{\mu^-}(v_i)$ is the least value among all other eccentricity. That is,

$$r_\mu(G) = e_\mu(v_i), \tag{7}$$

where $e_\mu(v_i) < e_\mu(v_j), \text{for some} v_i, v_j \in V$ and

$$\delta_\mu(v_i, v_j) = e_\mu(v_j) > e_\mu(v_i),\ \text{for some}\ v_i, v_j\ \in V. \tag{8}$$

Hence from equations (7) and (8), we have, $\delta_\mu(v_i, v_j) < r_\mu(G)$, for some $v_i, v_j \in V$, which is a contradiction to the fact that $\delta_\mu(v_i, v_j) \geq r_\mu(G)$, $\forall v_i, v_j \in V$. Hence G is a self centered graph. □

Theorem 32. *Let G be an $\mathcal{N}$-graph. If the graph G is a complete bipartite $\mathcal{N}$-graph then the complement of G is a self-centered $\mathcal{N}$-graph.*

Proof. A bipartite $\mathcal{N}$-graph G is said to be complete if

$$\mu_2(v_i, v_j) = \max(\mu_1(v_i), \mu_1(v_j)), \forall\ v_i \in V_1\ and\ v_j \in V_2$$

and

$$\mu_2(v_i, v_j) = 0, \forall\ v_i, v_j \in V_1\ or\ v_i, v_j \in V_2 \tag{9}$$

Now

$$\bar{\mu}_2(v_i, v_j) = \max(\mu_1(v_i), \mu_1(v_j)) - \mu_{2ij} \tag{10}$$

By using equation(9)

$$\bar{\mu}_2(v_i, v_j) = \max(\mu_1(v_i), \mu_1(v_j)), \forall\ v_i, v_j \in V_1, v_i, v_j \in V_2 \tag{11}$$

Hence from equations (9), (10) and (11), the complement of G has two components and each component is a complete $\mathcal{N}$-graph, which are self centered $\mathcal{N}$-graph. This completes the proof. □

CONCLUSIONS

Graph theory is rapidly moving into the mainstream of mathematics mainly because of its applications in diverse fields which include biochemistry (genomics), electrical engineering (communications networks and coding theory), computer science (algorithms and computations) and operations research (scheduling). It is known that fuzzy models give more precision, flexibility and compatibility to the system as compare to the classic models. In this paper, we have introduced the concept of self centered $\mathcal{N}$-graphs and have investigated some of their interesting properties. The concept of $\mathcal{N}$-graphs can be applied in various domains such as biochemistry, engineering, computer science and operations research.

REFERENCES

1. Jun, Y.B., K.J. Lee and S.Z. Song. 2009. Nideals of BCK/BCI-algerbas, Journal of the ChungcheongMathematical Society, 22: 417-437.
2. Rosenfeld, A. 1975. Fuzzy graphs, Fuzzy Sets and their Applications (L.A. Zadeh, K.S. Fu, M. Shimura, Eds.), Academic Press, New York, 77-95.
3. Kauffman, A. 1973. Introduction a la Theorie des Sous-emsembles Flous, Masson et Cie, Vol.1.
4. Bhattacharya, P. 1987. Some remarks on fuzzy graphs, Pattern Recognition Letter, 6: 297-302.
5. Akram, M. 2011. Bipolar fuzzy graphs, Information Sciences, 181: 5548-5564.
6. Akram,M. 2012. Interval-valued fuzzy line graphs, Neural Computing and Applications, 21:145-150.
7. Akram, M. and W.A. Dudek, 2011. Interval-valued fuzzy graphs, Computers Math. Applications, 61: 289-299.
8. Akram, M. and M.G. Karunambigai, 2012. Metric in bipolar fuzzy graphs,World Applied Sciences Journal, 14:1920-1927.

9. Akram, M. 2013. Bipolar fuzzy graphs with applications, Knowledge-Based Systems, 39: 1-8.

10. Buckley, F. 1989. Self-centered graphs, Graph Theory and Its Applications: East and West. Ann. New York Acad. Sci., 576:71-78.

11. Mordeson, J.N. and P.S. Nair, 2001. Fuzzy graphs and fuzzy hypergraphs, Physica Verlag, Heidelberg.

12. Sunitha, M. S. and K. Sameena, 2008. Characterization of g-self centered fuzzy graphs, The Journal of Fuzzy Mathematics, 16: 787-791.

13. Zadeh, L.A. 1965. Fuzzy sets, Information and Control, 8:338-353.

14. Zadeh, L.A. 1971. Similarity relations and fuzzy orderings, Information Sciences, 3:177-200