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$\mathcal{N}$-structures Applied to Graphs

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Abstract: In this paper, we introduce the notion of $\mathcal{N}$-graphs and describe methods of their construction. We prove that the isomorphism between $\mathcal{N}$-graphs is an equivalence relation (resp. partial order relation). We then introduce the concept of $\mathcal{N}$-line graphs and discuss some of their fundamental properties.

Key words: $\mathcal{N}$-graphs, isomorphism, $\mathcal{N}$-line graphs.

INTRODUCTION

A (crisp) set A in a universe X can be defined in the form of its characteristic function $\mu_A : X \to \{0, 1\}$ yielding the value 1 for elements belonging to the set A and the value 0 for elements excluded from the set A. The most of the generalization of the crisp set have been introduced on the unit interval $[0, 1]$ and they are consistent with the asymmetry observation. In other words, the generalization of the crisp set to fuzzy sets relied on spreading positive information that fit the crisp point $\{1\}$ into the interval $[0, 1]$. Because no negative meaning of information is suggested, we now feel a need to deal with negative information. To do so, we also feel a need to supply mathematical tool. To attain such object, Jun *et al.* [1] have introduced a new function which is called negative-valued function (briefly, $\mathcal{N}$-function) to deal with negative information that fit the crisp point $\{-1\}$ into the interval $[-1, 0]$, and constructed $\mathcal{N}$-structures. It is important to be able to deal with negative information. It is noted that positive information represents what is granted to be possible, while negative information represents what is considered to be impossible. As an example, let us consider the spatial relations. Human beings consider "left" and "right" as opposite directions. But this does not mean that one of them is the negation of the other. The semantics of "opposite" captures a notion of symmetry rather than a strict complementation. In particular, there may be positions which are considered neither to the right nor to the left of some reference object.

In 1975, Rosenfeld [2] discussed the concept of fuzzy graphs whose basic idea was introduced by Kauffmann [3] in 1973. The fuzzy relations between fuzzy sets were also considered by Rosenfeld and he developed the structure of fuzzy graphs obtaining analogs of several graph theoretical concepts. Bhattacharya [4] gave some remarks on fuzzy graphs. Akram *et al.* introduced the concepts of bipolar fuzzy graphs and interval-valued fuzzy line graphs [5-9]. In this paper, we introduce the notion of $\mathcal{N}$-graphs, describe methods of their construction. We prove that the isomorphism between $\mathcal{N}$-graphs is an equivalence relation (resp. partial order relation). We then introduce the concept of $\mathcal{N}$-line graphs and discuss some of their fundamental properties. We have used standard definitions and terminologies in this paper. For other notations, terminologies and applications not mentioned in the paper, the readers are referred to [10-14].

PRELIMINARIES

Recall that a *graph* is an ordered pair $G^* = (V, E)$, where V is the set of vertices of G^* and E is the set of edges of G^*. Two vertices x and y in an undirected graph G^* are said to be adjacent in G^* if $\{x, y\} = xy$ is an edge of G^*. A *simple graph* is an undirected graph that has no loops and no more than one edge between any two different vertices. A *subgraph* of a graph $G^* = (V, E)$ is a graph $H = (W, F)$, where $W \subseteq V$ and $F \subseteq E$. The *complementary graph* $\overline{G^*}$ of a simple graph has the same vertices as G^*. Two vertices are adjacent in $\overline{G^*}$ if and only if they are not adjacent in G^*. Consider the *Cartesian product* $G^* = G_1^* \times G_2^* = (V, E)$ of graphs G_1^* and G_2^*. Then $V = V_1 \times V_2$ and $E = \{(x, x_2)(x, y_2)|x_1 \in V_1, x_2y_2 \in E_2\} \cup \{(x_1, z)(y_1, z)|z \in V_2, x_1y_1 \in E_1\}$. Let $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ be two simple graphs. Then, the *composition* of graph G_1^* with G_2^* is denoted by $G_1^*[G_2^*] = (V_1 \times V_2, E^0)$, where $E^0 = E \cup \{(x_1, x_2)(y_1, y_2)|x_1y_1 \in E_1, x_2 \neq y_2\}$ and E is defined in $G_1^* \times G_2^*$. Note that $G_1^*[G_2^*] \neq G_2^*[G_1^*]$. The *union* of graphs G_1^* and G_2^* is defined as $G_1^* \cup G_2^* = (V_1 \cup V_2, E_1 \cup E_2)$. The *join* of G_1^* and G_2^* is the simple graph $G_1^* + G_2^* = (V_1 \cup V_2, E_1 \cup E_2 \cup E')$, where E' is the set of all edges joining the nodes of V_1 and V_2. In this construction it is assumed that $V_1 \cap V_2 \neq \emptyset$. An *isomorphism* of the graphs G_1^* and G_2^* is a bijection between the vertex sets of G_1^* and G_2^* such that any two vertices v_1 and v_2 of G_1^* are adjacent in G_1^* if and only if $f(v_1)$ and $f(v_2)$ are adjacent in G_2^*. If an isomorphism exists between two graphs, then the graphs are called isomorphic and we write $G_1^* \simeq G_2^*$. An *automorphism* of a graph is a graph isomorphism with itself, i.e., a mapping from the vertices of the given graph G^* back to vertices of G^* such that the resulting graph G^* is isomorphic with G^*. By an *intersection graph* of a graph $G^* = (V, E)$, we mean, a pair $P(S) = (S, T)$ where $S = \{S_1, S_2, \ldots, S_n\}$ is a family of distinct nonempty subsets of V and $T = \{S_iS_j \,|\, S_i, S_j \in S, S_i \cap S_j \neq \emptyset, i \neq j\}$. It is well know that every graph is an intersection graph. By a *line graph* of a graph $G^* = (V, E)$, we mean, a pair $L(G^*) = (Z, W)$ where $Z = \{\{x\} \cup \{u_x, v_x\} \,|\, x \in E, u_x, v_x \in V, x = u_xv_x\}$ and $W = \{S_xS_y \,|\, S_x \cap S_y \neq \emptyset, x, y \in E, x \neq y\}$, and $S_x = \{x\} \cup \{u_x, v_x\}$, $x \in E$. It is reported in the literature that the line graph is an intersection graph. Denote by $\mathcal{F}(X, [-1, 0])$ the collection of functions from a nonempty set X to $[-1, 0]$. We say that an element of $\mathcal{F}(X, [-1, 0])$ is a *negative-valued function* from X to $[-1, 0]$ (briefly, $\mathcal{N}$*-function* on X). By an $\mathcal{N}$*-structure* we mean an ordered pair (X, μ) of X and an $\mathcal{N}$-function μ on X. By an $\mathcal{N}$*-relation* on X we mean an $\mathcal{N}$-function ν on $X \times X$ satisfying the following inequality:

$$(\forall x, y \in X)(\nu(x, y) \geq \max\{\mu(x), \mu(y)\}), \qquad (1)$$

where $\mu \in \mathcal{F}(X, [-1, 0])$. Throughout this paper, G^* will be a crisp graph, and G a $\mathcal{N}$-graph.

$\mathcal{N}$-STRUCTURES APPLIED TO GRAPHS

Definition 1. *An $\mathcal{N}$-graph with an underlying set V is defined to be a pair $G = (\mu, \nu)$ where μ is an $\mathcal{N}$-function in V and ν is an $\mathcal{N}$-function in $E \subseteq V \times V$ such that*

$$\nu(\{x, y\}) \geq \max(\mu(x), \mu(y))$$

for all $\{x, y\} \in E$. We call μ the $\mathcal{N}$-vertex function of V, ν the $\mathcal{N}$-edge function of E, respectively. Note that ν is a symmetric $\mathcal{N}$-relation on μ. We use the notation xy for an element $\{x, y\}$ of E. Thus, $G = (\mu, \nu)$ is an $\mathcal{N}$-graph of $G^* = (V, E)$ if

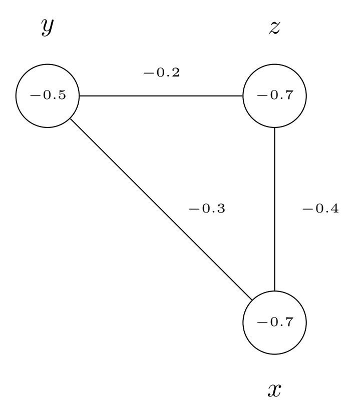
$$\nu(xy) \geq \max(\mu(x), \mu(y)) \quad \text{for all } \; xy \in E.$$

Definition 2. *Let $G = (\mu, \nu)$ be an $\mathcal{N}$-graph. The order of an $\mathcal{N}$-graph is defined by $O(G) = \sum_{x \in V} \mu(x)$. The degree of a vertex x in G is defined by $\deg(x) = \sum_{xy \in E} \nu(xy)$.*

Example 3. *Consider a graph $G^* = (V, E)$ such that $V = \{x, y, z\}$, $E = \{xy, yz, zx\}$. Let μ be an $\mathcal{N}$-function of V and let ν be an $\mathcal{N}$-function of $E \subseteq V \times V$ defined by*

	x	y	z
μ	-0.7	-0.5	-0.7

	xy	yz	zx
ν	-0.3	-0.2	-0.4



G

(i) *By routine computations, it is easy to see that $G = (\mu, \nu)$ is an $\mathcal{N}$-graph of G^*.*
(ii) *Order of an $\mathcal{N}$-graph= $O(G)$=-1.9.*
(iii) *Degree of each vertex in G is*

$$\deg(x) = -0.7, \; \deg(y) = -0.5, \; \deg(z) = -0.6.$$

Definition 4. *Let μ_1 and μ_2 be $\mathcal{N}$-functions of V_1 and V_2 and let ν_1 and ν_2 be $\mathcal{N}$-functions of E_1 and E_2, respectively. The Cartesian product of two $\mathcal{N}$-graphs G_1 and G_2 of the graphs G_1^* and G_2^* is denoted by $G_1 \times G_2 = (\mu_1 \times \mu_2, \nu_1 \times \nu_2)$ and is defined as follows:*

- $(\mu_1 \times \mu_2)(x_1, x_2) = \max(\mu_1(x_1), \mu_2(x_2))$ *for all* $(x_1, x_2) \in V$,

- $(\nu_1 \times \nu_2)((x, x_2)(x, y_2)) = \max(\mu_1(x), \nu_2(x_2y_2))$ *for all* $x \in V_1$, *for all* $x_2y_2 \in E_2$,
- $(\nu_1 \times \nu_2)((x_1, z)(y_1, z)) = \max(\nu_1(x_1y_1), \mu_2(z))$ *for all* $z \in V_2$, *for all* $x_1y_1 \in E_1$.

Definition 5. *Let* G_1 *and* G_2 *be two* $\mathcal{N}$*- graphs. The degree of a vertex in* $G_1 \times G_2$ *can be defined as follows: for any* $(x_1, x_2) \in V_1 \times V_2$,

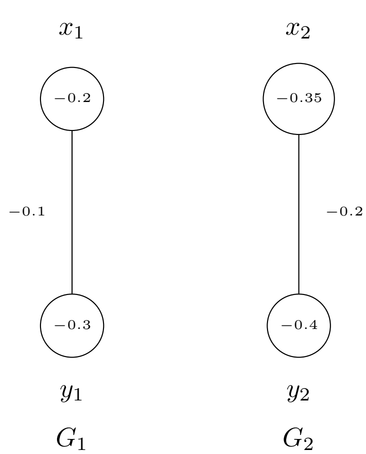
$$\begin{aligned} d_{G_1\times G_2}(x_1,x_2) &= \sum_{(x_1,x_2)(y_1,y_2)\in E} (\nu_1\times\nu_2)(x_1,x_2)(y_1,y_2) \\ &= \sum_{x_1=y_1=x, x_2y_2\in E_2} \max(\mu_1(x),\nu_2(x_2y_2)) \\ &+ \sum_{x_2=y_2=z, x_1y_1\in E_1} \max(\mu_2(z),\nu_1(x_1y_1)). \end{aligned}$$

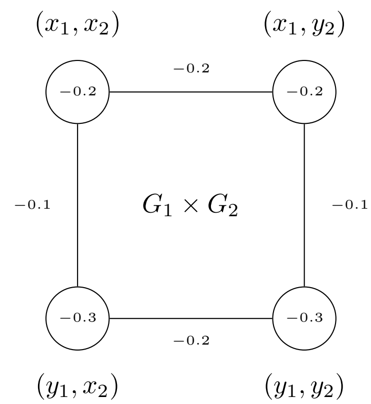
Example 6. *Let* $G_1^* = (V_1, E_1)$ *and* $G_2^* = (V_2, E_2)$ *be two graphs, where* $V_1 = \{x_1, y_1\}$ *and* $V_2 = \{x_2, y_2\}$ *are underlying sets. Let* μ_1 *and* μ_2 *be* $\mathcal{N}$*-functions of* V_1 *and* V_2 *and let* ν_1 *and* ν_2 *be* $\mathcal{N}$*-functions of* E_1 *and* E_2*, respectively. We define* $\mu_1 : V_1 \to [-1, 0]$, $\mu_2 : V_2 \to [-1, 0]$, $\nu_1 : E_1 \to [-1, 0]$ *and* $\nu_2 : E_2 \to [-1, 0]$ *by*

$$\mu_1(x_1) = -0.2,\ \mu_1(y_1) = -0.3,$$
$$\mu_2(x_2) = -0.35,\ \mu_2(y_2) = -0.4,$$
$$\nu_1(x_1y_1) = -0.1,\ \nu_2(x_2y_2) = -0.2.$$

(i) It is easy to see that $G_1 = (\mu_1, \nu_1)$ *and* $G_2 = (\mu_2, \nu_2)$ *are* $\mathcal{N}$*-graphs of* G_1^* *and* G_2^**, respectively. Routine computations give*

$$(\mu_1\times\mu_2)(x_1,x_2) = -0.2,\ (\mu_1\times\mu_2)(x_1,y_2) = -0.2,$$
$$(\mu_1\times\mu_2)(y_1,x_2) = -0.3,\ (\mu_1\times\mu_2)(y_1,y_2) = -0.3,$$
$$(\nu_1\times\nu_2)((x_1,x_2)(x_1,y_2)) = -0.2,$$
$$(\nu_1\times\nu_2)((x_1,x_2)(y_1,x_2)) = -0.1,$$
$$(\nu_1\times\nu_2)((y_1,x_2)(y_1,y_2)) = -0.2,$$
$$(\nu_1\times\nu_2)((x_1,y_2)(y_1,y_2)) = -0.1.$$





Clearly, $G_1 \times G_2$ *is an* $\mathcal{N}$*- graph of* $G_1^* \times G_2^*$.
(ii) Routine computations give degree of each vertex in $G_1 \times G_2$ *as*

$$d_{G_1\times G_2}(x_1,x_2) = -0.3,\ d_{G_1\times G_2}(x_1,y_2) = -0.3,$$
$$d_{G_1\times G_2}(y_1,x_2) = -0.3,\ d_{G_1\times G_2}(y_1,y_2) = -0.3.$$

Definition 7. *Let* μ_1 *and* μ_2 *be* $\mathcal{N}$*-functions of* V_1 *and* V_2 *and let* ν_1 *and* ν_2 *be* $\mathcal{N}$*-functions of* E_1 *and* E_2*, respectively. The composition of two* $\mathcal{N}$*-graphs* G_1 *and* G_2 *of the graphs* G_1^* *and* G_2^* *is denoted by* $G_1[G_2] = (\mu_1 \circ \mu_2, \nu_1 \circ \nu_2)$ *and is defined as follows:*

- $(\mu_1 \circ \mu_2)(x_1, x_2) = \max(\mu_1(x_1), \mu_2(x_2))$ *for all* $(x_1, x_2) \in V$,
- $(\nu_1 \circ \nu_2)((x, x_2)(x, y_2)) = \max(\mu_1(x), \nu_2(x_2y_2))$ *for all* $x \in V_1$, *for all* $x_2y_2 \in E_2$,
- $(\nu_1 \circ \nu_2)((x_1, z)(y_1, z)) = \max(\nu_1(x_1y_1), \mu_2(z))$ *for all* $z \in V_2$, *for all* $x_1y_1 \in E_1$.
- $(\nu_1\circ\nu_2)((x_1, x_2)(y_1, y_2)) = \max(\mu_2(x_2), \mu_2(y_2), \nu_1(x_1y_1))$ *for all* $z \in V_2$, *for all* $(x_1, x_2)(y_1, y_2) \in E^0 - E$.

Note that $\mu_1 \circ \mu_2 = \mu_1 \times \mu_2$ *on* V *and* $\nu_1 \circ \nu_2 = \nu_1 \times \nu_2$ *on* E.

Definition 8. *Let* G_1 *and* G_2 *be two* $\mathcal{N}$*-graphs. The degree of a vertex in* $G_1[G_2]$ *can be defined as follows: for any* $(x_1, x_2) \in V_1 \times V_2$,

$$\begin{aligned} d_{G_1[G_2]}(x_1,x_2) &= \sum_{(x_1,x_2)(y_1,y_2)\in E} (\nu_1\circ\nu_2)(x_1,x_2)(y_1,y_2) \\ &= \sum_{x_1=y_1=x, x_2y_2\in E_2} \max(\mu_1(x),\nu_2(x_2y_2)) \\ &+ \sum_{x_2=y_2=z, x_1y_1\in E_1} \max(\mu_2(z),\nu_1(x_1y_1)) \\ &+ \sum_{x_2\neq y_2, x_1y_1\in E_1} \max(\mu_2(x_2),\nu_1(x_1y_1)). \end{aligned}$$

Example 9. *Let $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ be two graphs, where $V_1 = \{x_1, y_1\}$ and $V_2 = \{x_2, y_2\}$ be underlying sets. Let μ_1 and μ_2 be $\mathcal{N}$-functions of V_1 and V_2 and let ν_1 and ν_2 be $\mathcal{N}$-functions of E_1 and E_2, respectively. We define $\mu_1 : V_1 \to [-1, 0]$, $\mu_2 : V_2 \to [-1, 0]$, $\nu_1 : E_1 \to [-1, 0]$ and $\nu_2 : E_2 \to [-1, 0]$ by*

$$\mu_1(x_1) = -0.2,\ \mu_1(y_1) = -0.3,$$

$$\mu_2(x_2) = -0.35,\ \mu_2(y_2) = -0.4,$$

$$\nu_1(x_1y_1) = -0.35,\ \nu_2(x_2, y_2) = -0.5.$$

(i) It is easy to see that $G_1 = (\mu_1, \nu_1)$ and $G_2 = (\mu_2, \nu_2)$ are $\mathcal{N}$-graphs of G_1^* and G_2^*, respectively. Routine computations give

$$(\mu_1 \circ \mu_2)(x_1, x_2) = -0.2,\ (\mu_1 \circ \mu_2)(x_1, y_2) = -0.2,$$

$$(\mu_1 \circ \mu_2)(y_1, x_2) = -0.3,\ (\mu_1 \circ \mu_2)(y_1, y_2) = -0.3,$$

$$(\nu_1 \circ \nu_2)((x_1, x_2)(x_1, y_2)) = -0.2,$$

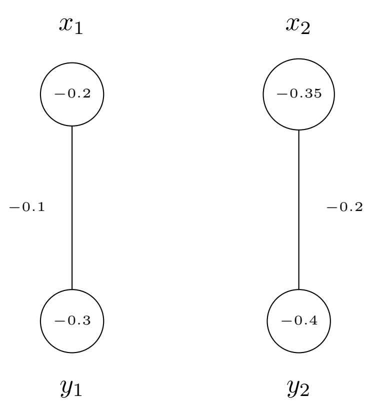
$$(\nu_1 \circ \nu_2)((x_1, x_2)(y_1, x_2)) = -0.1,$$

$$(\nu_1 \circ \nu_2)((y_1, x_2)(y_1, y_2)) = -0.2,$$

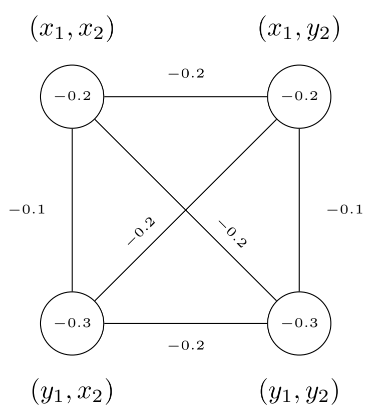
$$(\nu_1 \circ \nu_2)((x_1, y_2)(y_1, y_2)) = -0.1.$$

$$(\nu_1 \circ \nu_2)((x_1, x_2)(y_1, y_2)) = -0.2,$$

$$(\nu_1 \circ \nu_2)((y_1, x_2)(x_1, y_2)) = -0.2.$$



G_1 G_2



$G_1[G_2]$

Clearly, $G_1[G_2]$ is an $\mathcal{N}$- graph of $G_1^*[G_2^*]$.
(ii) Routine computations give degree of each vertex in $G_1[G_2]$ as

$$d_{G_1[G_2]}(x_1, x_2) = -0.5,\ d_{G_1[G_2]}(x_1, y_2) = -0.5,$$

$$d_{G_1[G_2]}(y_1, x_2) = -0.5,\ d_{G_1[G_2]}(y_1, y_2) = -0.5.$$

Definition 10. *Let μ_1 and μ_2 be $\mathcal{N}$-functions of V_1 and V_2 and let ν_1 and ν_2 be $\mathcal{N}$-functions of E_1 and E_2, respectively. Then union of two $\mathcal{N}$-graphs G_1 and G_2 of the graphs G_1^* and G_2^* is denoted by $G_1 \cup G_2 = (\mu_1 \cup \mu_2, \nu_1 \cup \nu_2)$ and is defined as follows:*

(A) $(\mu_1 \cup \mu_2)(x) = \mu_1(x)$ *if* $x \in V_1 \cap \overline{V_2}$,
$(\mu_1 \cup \mu_2)(x) = \mu_2(x)$ *if* $x \in V_2 \cap \overline{V_1}$,
$(\mu_1 \cup \mu_2)(x) = \max(\mu_1(x), \mu_2(x))$ *if* $x \in V_1 \cap V_2$.

(B) $(\nu_1 \cup \nu_2)(xy) = \nu_1(xy)$ *if* $xy \in E_1 \cap \overline{E_2}$,
$(\nu_1 \cup \nu_2)(xy) = \nu_2(xy)$ *if* $xy \in E_2 \cap \overline{E_1}$,
$(\nu_1 \cup \nu_2)(xy) = \max(\nu_1(xy), \nu_2(xy))$ *if* $xy \in E_1 \cap E_2$.

Definition 11. *Let G_1 and G_2 be two $\mathcal{N}$-graphs. The degree of a vertex in $G_1 \cup G_2$ can be defined as follows:*

Case 1: *When $x \in V_1$ or $x \in V_2$ but not in both.*
If $x \in V_1$, then $d_{G_1 \cup G_2}(x) = \sum_{xy \in E_1} \nu_1(xy)$,
If $x \in V_2$, then $d_{G_1 \cup G_2}(x) = \sum_{xy \in E_2} \nu_2(xy)$.

Case 2: *When $x \in V_1 \cap V_2$ but no edge incident at x lies in $E_1 \cap E_2$.*
$d_{G_1 \cup G_2}(x) = d_{G_1}(x) + d_{G_2}(x)$.

Case 3: *When $x \in V_1 \cup V_2$ and some edges incident at x lies in $E_1 \cap E_2$.*
$d_{G_1 \cup G_2}(x) = d_{G_1}(x) + d_{G_2}(x)$- $\sum_{xy \in E_1 \cap E_2} \max(\nu_1(xy), \nu_2(xy)$.

Example 12. *Let $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ be two graphs, where $V_1 = \{a, b, c, d, e\}$ and $V_2 = \{a, b, c, d, f\}$ be underlying sets. Let μ_1 and μ_2 be $\mathcal{N}$-functions of V_1 and V_2 and let ν_1 and ν_2 be $\mathcal{N}$-functions of E_1 and E_2, respectively. We define $\mu_1 : V_1 \to [-1, 0]$, $\mu_2 : V_2 \to [-1, 0]$, $\nu_1 : E_1 \to [-1, 0]$ and $\nu_2 : E_2 \to [-1, 0]$ by*

$$\mu_1(a) = -0.2,\ \mu_1(b) = -0.1,$$

$$\mu_1(c) = -0.3,\ \mu_1(d) = -0.3, \mu_1(e) = -0.4,$$

$$\mu_2(a) = -0.1,\ \mu_2(b) = -0.2,\ \mu_2(c) = -0.3,$$

$$\mu_2(d) = -0.5, \mu_2(f) = -0.4,$$

$$\nu_1(ab) = -0.1,\ \nu_1(bc) = -0.1,\ \nu_1(ce) = -0.2,$$

$$\nu_1(be) = -0.1,\ \nu_1(ad) = -0.1,\ \nu_1(de) = -0.2,$$
$$\nu_2(ab) = -0.1,\ \nu_2(bc) = -0.1,$$
$$\nu_2(cf) = -0.2,\ \nu_2(bd) = -0.2,\ \nu_2(bf) = -0.1.$$

It is easy to see that $G_1 = (\mu_1, \nu_1)$ and $G_2 = (\mu_2, \nu_2)$ are $\mathcal{N}$-graphs of G_1^* and G_2^*, respectively. Routine computations give

$$(\mu_1 \cup \mu_2)(a) = -0.1,\ (\mu_1 \cup \mu_2)(b) = -0.1,$$
$$(\mu_1 \cup \mu_2)(c) = -0.3,\ (\mu_1 \cup \mu_2)(d) = -0.3,$$
$$(\mu_1 \cup \mu_2)(e) = -0.4,\ (\mu_1 \cup \mu_2)(f) = -0.4,$$
$$(\nu_1 \cup \nu_2)(ab) = -0.1,\ (\nu_1 \cup \nu_2)(bc) = -0.1,$$
$$(\nu_1 \cup \nu_2)(ce) = -0.2,\ (\nu_1 \cup \nu_2)(be) = -0.1,$$
$$(\nu_1 \cup \nu_2)(ad) = -0.1,\ (\nu_1 \cup \nu_2)(de) = -0.2,$$
$$(\nu_1 \cup \nu_2)(bd) = -0.2, (\nu_1 \cup \nu_2)(de) = -0.1.$$

Clearly, $(\mu_1 \cup \mu_2, \nu_1 \cup \nu_2)$ is an $\mathcal{N}$- graph of $G_1^* \cup G_2^*$.

Definition 13. *Let μ_1 and μ_2 be $\mathcal{N}$-functions of V_1 and V_2 and let ν_1 and ν_2 be $\mathcal{N}$-functions of E_1 and E_2, respectively. Then join of two $\mathcal{N}$-graphs G_1 and G_2 of the graphs G_1^* and G_2^* is denoted by $G_1 + G_2 = (\mu_1 + \mu_2, \nu_1 + \nu_2)$ and is defined as follows:*

- $(\mu_1 + \mu_2)(x) = (\mu_1 \cup \mu_2)(x)$ *if* $x \in V_1 \cup V_2$,
- $(\nu_1 + \nu_2)(xy) = (\nu_1 \cup \nu_2)(xy) = \nu_1(xy)$ *if* $xy \in E_1 \cup E_2$,
- $(\nu_1 + \nu_2)(xy) = \max(\mu_1(x), \mu_2(y))$ *if* $xy \in E'$.

Proposition 14. *If G_1 and G_2 are the $\mathcal{N}$-graphs, then $G_1 \times G_2$, $G_1[G_2]$, $G_1 \cup G_2$ and $G_1 + G_2$ are $\mathcal{N}$-graphs.*

We formulate the following characterizations.

Proposition 15. *Let $G_1 = (\mu_1, \nu_1)$ and $G_2 = (\mu_2, \nu_2)$ be $\mathcal{N}$-graphs of the graphs G_1^* and G_2^* and let $V_1 \cap V_2 = \emptyset$. Then union $G_1 \cup G_2 = (\mu_1 \cup \mu_2, \nu_1 \cup \nu_2)$ is an $\mathcal{N}$-graph of G^* if and only if $G_1 = (\mu_1, \nu_1)$ and $G_2 = (\mu_2, \nu_2)$ are $\mathcal{N}$-graphs of the graphs G_1^* and G_2^*, respectively.*

Proof. Suppose that $G_1 \cup G_2$ is an $\mathcal{N}$-graph. Let $xy \in E_1$. Then $xy \notin E_2$ and $x, y \in V_1 - V_2$. Thus

$$\begin{aligned}\nu_1(xy) &= (\nu_1 \cap \nu_2)(xy)\\ &\geq \max((\mu_1 \cap \mu_2)(x), (\mu_1 \cap \mu_2)(y))\\ &= \max(\mu_1(x), \mu_1(y)).\end{aligned}$$

This shows that $G_1 = (\mu_1, \nu_1)$ is an $\mathcal{N}$-graph. Similarly, we can show that $G_2 = (\mu_2, \nu_2)$ is an $\mathcal{N}$-graph. The converse part is obvious. □

As a consequence of above propositions, we obtain

Proposition 16. *Let $G_1 = (\mu_1, \nu_1)$ and $G_2 = (\mu_2, \nu_2)$ be $\mathcal{N}$-graphs of the graphs G_1^* and G_2^* and let $V_1 \cap V_2 = \emptyset$. Then join $G_1 + G_2 = (\mu_1 + \mu_2, \nu_1 + \nu_2)$ is an $\mathcal{N}$-graph of G^* if and only if $G_1 = (\mu_1, \nu_1)$ and $G_2 = (\mu_2, \nu_2)$ are $\mathcal{N}$-graphs of the graphs G_1^* and G_2^*, respectively.*

We now discuss isomorphism of $\mathcal{N}$-graphs.

Definition 17. *Let $G_1 = (\mu_1, \nu_1)$ and $G_2 = (\mu_2, \nu_2)$ be $\mathcal{N}$-graphs. A homomorphism $f : G_1 \to G_2$ is a mapping $f : V_1 \to V_2$ such that*

(i) $\mu_1(x_1) \geq \mu_2(f(x_1))$,

(ii) $\nu_1(x_1y_1) \geq \nu_2(f(x_1)f(y_1))$

for all $x_1 \in V_1$, $x_1y_1 \in E_1$. A bijective homomorphism with the property

(iii) $\mu_1(x_1) = \mu_2(f(x_1))$

is called a strong isomorphism. A strong isomorphism preserves the weights of the nodes but not necessarily the weights of the arcs. A bijective homomorphism preserving the weights of the arcs but not necessarily the weights of nodes, i.e., a bijective homomorphism $f : G_1 \to G_2$ such that

(iv) $\nu_1(x_1y_1) = \nu_2(f(x_1)f(y_1))$

for all $x_1y_1 \in V_1$ is called a strong co-isomorphism. A bijective mapping $f : G_1 \to G_2$ satisfying (iii) and (iv) is called an isomorphism.

Proposition 18. *An isomorphism between $\mathcal{N}$-graphs is an equivalence relation.*

Proof. The reflexivity and symmetry are obvious. To prove the transitivity, we let $f : V_1 \to V_2$ and $g : V_2 \to V_3$ be the isomorphisms of G_1 onto G_2 and G_2 onto G_3, respectively. Then $g \circ f : V_1 \to V_3$ is a bijective map from V_1 to V_3, where $(g \circ f)(x_1) = g(f(x_1))$ for all $x_1 \in V_1$. Since a map $f : V_1 \to V_2$ defined by $f(x_1) = x_2$ for $x_1 \in V_1$ is an isomorphism, so we have

$$\mu_1(x_1) = \mu_2(f(x_1)) = \mu_2(x_2) \text{ for all } x_1 \in V_1 \ \cdots(A),$$

$$\begin{aligned}\nu_1(x_1y_1) &= \nu_2(f(x_1)f(y_1))\\ &= \nu_2(x_2y_2) \text{ for all } x_1y_1 \in E_1 \ \cdots(B).\end{aligned}$$

Since a map $g : V_2 \to V_3$ defined by $g(x_2) = x_3$ for $x_2 \in V_2$ is an isomorphism, so

$$\mu_2(x_2) = \mu_3(g(x_2)) = \mu_3(x_3) \text{ for all } x_2 \in V_2 \ \cdots(C),$$

$$\begin{aligned}\nu_2(x_2y_2) &= \nu_3(g(x_2)g(y_2)) \\ &= \nu_3(x_3y_3) \text{ for all } x_2y_2 \in E_2 \quad \cdots (D).\end{aligned}$$

From (A), (C) and $f(x_1) = x_2$, $x_1 \in V_1$, we have

$$\begin{aligned}\mu_1(x_1) &= \mu_2(f(x_1)) = \mu_2(x_2) \\ &= \mu_3(g(x_2)) = \mu_3(g(f(x_1))), \quad \text{for all } x_1 \in V_1,\end{aligned}$$

From (B) and (D), we have

$$\begin{aligned}\nu_1(x_1y_1) &= \nu_2(f(x_1)f(y_1)) = \nu_2(x_2y_2) \\ &= \nu_3(g(x_2)g(y_2)) \\ &= \nu_3(g(f(x_1))g(f(y_1)))\end{aligned}$$

for all $x_1y_1 \in E_1$.

Therefore, $g \circ f$ is an isomorphism between G_1 and G_3. This completes the proof. □

Proposition 19. *A weak isomorphism (co-isomorphism) between $\mathcal{N}$-graphs is a partial ordering relation.*

Proof. The reflexivity and transitivity are obvious. To prove the anti symmetry, we let $f : V_1 \to V_2$ be a strong isomorphism of G_1 onto G_2. Then f is a bijective map defined by $f(x_1) = x_2$ for all $x_1 \in V_1$
satisfying

$$\mu_1(x_1) = \mu_2(f(x_1)) \text{ for all } x_1 \in V_1,$$

$$\nu_1(x_1y_1) \geq \nu_2(f(x_1)f(y_1)) \text{ for all } x_1y_1 \in E_1 \cdots (E).$$

Let $g : V_2 \to V_1$ be a strong isomorphism of G_2 onto G_1. Then g is a bijective map defined by $g(x_2) = x_1$ for all $x_2 \in V_2$
satisfying

$$\mu_2(x_2) = \mu_1(g(x_2)) \text{ for all } x_2 \in V_2,$$

$$\nu_2(x_2y_2) \geq \nu_1(g(x_2)g(y_2)) \text{ for all } x_2y_2 \in E_2 \quad \cdots (F).$$

The inequalities (E) and (F) hold on the finite sets V_1 and V_2 only when G_1 and G_2 have the same number of edges and the corresponding edges have same weight. Hence G_1 and G_2 are identical. Therefore, $g \circ f$ is a strong isomorphism between G_1 and G_3. This completes the proof. □

Definition 20. *The complement of a weak negative-valued fuzzy graph $G = (\mu, \nu)$ of $G^* = (V, E)$ is a weak $\mathcal{N}$-graph $\overline{G} = (\overline{\mu}, \overline{\nu})$ on $\overline{G^*}$, is defined by*

(i)

$$\overline{V} = V,$$

(ii)

$$\overline{\mu(x)} = \mu(x) \qquad \text{for all } x \in V,$$

(iii)

$$\overline{\nu}(xy) = \begin{cases} 0 & \text{if } \nu(xy) > 0, \\ \max(\nu(x), \nu(y)) & \text{if if } \nu(xy) = 0. \end{cases}$$

Definition 21. *An $\mathcal{N}$-graph G is called self complementary if $\overline{G} \approx G$.*

The following propositions are obvious.

Proposition 22. *Let G be a self complementary $\mathcal{N}$-graph. Then*

$$\sum_{x \neq y} \nu(xy) = \frac{1}{2} \sum_{x \neq y} \max(\mu(x), \mu(y)).$$

Proposition 23. *Let G be an $\mathcal{N}$-graph. If $\nu(xy) = \max(\mu(x), \mu(y))$ for all x, $y \in V$, then G is self complementary.*

Proposition 24. *Let G_1 and G_2 be $\mathcal{N}$-graphs. Then $G_1 \cong G_2$ if and only if $\overline{G}_1 \cong \overline{G}_2$.*

Proof. Assume that G_1 and G_2 are isomorphic, there exists a bijective map $f : V_1 \to V_2$ satisfying

$$\nu_1(x) = \mu_2(f(x)) \text{ for all } x \in V_1,$$

$$\nu_1(xy) = \mu_2(f(x)f(y)) \text{ for all } xy \in E_1.$$

By definition of complement, we have

$$\overline{\nu}_1(xy) = \max(\mu_1(x), \mu_1(y) = \max(\mu_2(f(x)),$$

$$\mu_2(f(y))) = \overline{\mu}_2(f(x)f(y)) \text{ for all } xy \in E_1.$$

Hence $\overline{G}_1 \cong \overline{G}_2$. The proof of converse part is straightforward. This completes the proof. □

Proposition 25. *Let G_1 and G_2 be $\mathcal{N}$-graphs. If there is a strong isomorphism between G_1 and G_2, then there is a strong isomorphism between $\overline{G}_1$ and $\overline{G}_2$.*

Proof. Let f be a strong isomorphism between G_1 and G_2, then $f : V_1 \to V_2$ is a bijective map that satisfies $f(x_1) = x_2$ for all $x_1 \in V_1$,

$$\mu_1(x_1) = \mu_2(f(x_1)) \text{ for all } x_1 \in V_1,$$

$$\mu_1(x_1y_1) \geq \mu_2(f(x_1)f(y_1)) \text{ for all } x_1y_1 \in E_1.$$

Since $f : V_1 \to V_2$ is a bijective map, $f^{-1} : V_2 \to V_1$ is also bijective map such that $f^{-1}(x_2) = x_1$ for all $x_2 \in V_2$. Thus

$$\mu_1(f^{-1}(x_2)) = \mu_2(x_2) \text{ for all } x_2 \in V_2.$$

By definition of complement, we have

$$\begin{aligned}\overline{\nu}_1(x_1y_1) &= \max(\mu 1(x_1), \mu_1(y_1)) \\ &\geq \max(\mu_2(f(x_2)), \mu_2(f(y_2))) \\ &= \max(\mu_2(x_2), \mu_2(y_2)) \\ &= \overline{\nu}_2(x_2y_2).\end{aligned}$$

Thus, $f^{-1}: V_2 \to V_1$ is a bijective map which is a strong isomorphism between G_1 and $\overline{G}_2$. This ends the proof. □

The following Proposition is obvious.

Proposition 26. *Let G_1 and G_2 be $\mathcal{N}$-graphs. If there is a co-strong isomorphism between G_1 and G_2, then there is a homomorphism between $\overline{G}_1$ and $\overline{G}_2$.*

We now discuss $\mathcal{N}$-line graphs.

Definition 27. *Let $P(S) = (S, T)$ be an intersection graph of a simple graph $G^* = (V, E)$. Let $G = (\mu_1, \nu_1)$ be an $\mathcal{N}$-graph of G^*. We define an $\mathcal{N}$-intersection graph $P(G) = (\mu_2, \nu_2)$ of $P(S)$ as follows:*

(1) *μ_2 and ν_2 are $\mathcal{N}$-functions of S and T, respectively,*

(2) $\mu_2(S_i) = \mu_1(v_i)$,

(3) $\nu_2(S_iS_j) = \nu_2(v_iv_j)$

for all $S_i, S_j \in S$, $S_iS_j \in T$. That is, any $\mathcal{N}$-graph of $P(S)$ is called an $\mathcal{N}$-intersection graph.

The following Proposition is obvious.

Proposition 28. *Let $G = (\mu_1, \nu_1)$ be an $\mathcal{N}$-graph of G^*. Then*

- *$P(G) = (\mu_2, \nu_2)$ is an $\mathcal{N}$-graph of $P(S)$,*
- *$G \simeq P(G)$.*

This Proposition shows that any $\mathcal{N}$-graph is isomorphic to an $\mathcal{N}$-intersection graph.

Definition 29. *Let $L(G^*) = (Z, W)$ be a line graph of a simple graph $G^* = (V, E)$. Let $G = (\mu_1, \nu_1)$ be an $\mathcal{N}$- graph of G^*. We define an $\mathcal{N}$-line graph $L(G) = (\mu_2, \nu_2)$ of G as follows:*

(4) *μ_2 and ν_2 are $\mathcal{N}$-functions of Z and W, respectively,*

(5) $\mu_2(S_x) = \nu_1(x) = \nu_1(u_xv_x)$,

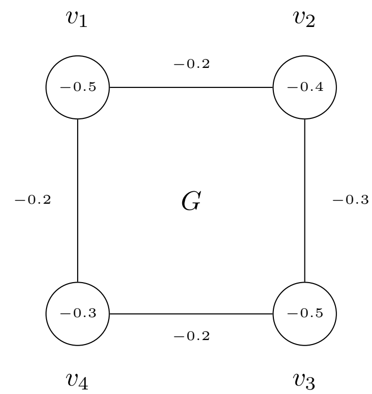
(6) $\nu_2(S_xS_y) = \max(\nu_1(x), \nu_1(y))$

for all $S_x, S_y \in Z$, $S_xS_y \in W$.

Example 30. *Consider a graph $G^* = (V, E)$ such that $V = \{v_1, v_2, v_3, v_4\}$ and $E = \{x_1 = v_1v_2,\ x_2 = v_2v_3,\ x_3 = v_3v_4,\ x_4 = v_4v_1\}$. Let μ_1 be an $\mathcal{N}$-function of V and let ν_1 be an $\mathcal{N}$-functions of E defined by*

	v_1	v_2	v_3	v_4
μ_1	-0.5	-0.4	-0.5	-0.3

	x_1	x_2	x_3	x_4
ν_1	-0.2	-0.3	-0.2	-0.2



By routine computations, it is easy to see that G is an $\mathcal{N}$-graph.
Consider a line graph $L(G^*) = (Z, W)$ such that

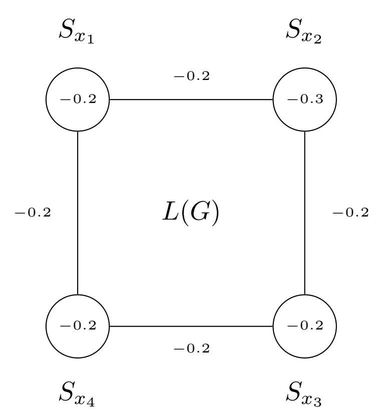
$$Z = \{S_{x_1}, S_{x_2}, S_{x_3}, S_{x_4}\}$$

and

$$W = \{S_{x_1}S_{x_2},\ S_{x_2}S_{x_3},\ S_{x_3}S_{x_4},\ S_{x_4}S_{x_1}\}.$$

Let μ_2 and ν_2 be $\mathcal{N}$-functions on Z and W, respectively. Then, by routine computations, we have

$$\mu_2(S_{x_1}) = -0.2,\ \mu_2(S_{x_2}) = -0.3,$$
$$\mu_2(S_{x_3}) = -0.2,\ \mu_2(S_{x_4}) = -0.2.$$
$$\nu_2(S_{x_1}S_{x_2}) = -0.2,\ \nu_2(S_{x_2}S_{x_3}) = -0.2,$$
$$\nu_2(S_{x_3}S_{x_4}) = -0.2,\ \nu_2(S_{x_4}S_{x_1}) = -0.2.$$



By routine computations, it is clear that $L(G)$ is an $\mathcal{N}$-line graph.

The following propositions are obvious.

Proposition 31. *$L(G)$ is an $\mathcal{N}$-line graph corresponding to $\mathcal{N}$-graph G.*

Proposition 32. *If $L(G)$ is an $\mathcal{N}$-line graph of $\mathcal{N}$-graph G. Then $L(G^*)$ is the line graph of G^*.*

Proposition 33. *$L(G)$ is an $\mathcal{N}$-line graph of some $\mathcal{N}$-graph G if and only if*

$$\nu_2(S_xS_y) = \max(\mu_2(S_x), \mu_2(S_y)) \text{ for all } \mathrm{S_xS_y} \in \mathrm{W}.$$

Proof. Assume that $\nu_2(S_xS_y) = \max(\mu_2(S_x), \mu_2(S_y))$ for all $S_x\ S_y \in W$. We define $\mu_1(x) = \mu_2(S_x)$ for all $x \in E$. Then

$$\nu_2(S_xS_y) = \max(\mu_2(S_x), \mu_2(S_y)) = \max(\mu_1(x), \mu_1(y)).$$

An $\mathcal{N}$-function (μ_1, ν_1) that yields that the property

$$\nu_1(xy) \geq \max(\mu_1(x), \mu_1(y))$$

will suffice. The converse part is obvious. □

Proposition 34. *$L(G)$ is an $\mathcal{N}$-line graph if and only if $L(G^*)$ is a line graph and*

$$\nu_2(uv) = \max(\mu_2(u), \mu_2(v)) \text{ for all uv} \in \mathrm{W}.$$

Proposition 35. *Let G_1 and G_2 be $\mathcal{N}$-graphs. If f is a strong isomorphism of G_1 onto G_2, then f is an isomorphism of G_1^* onto G_2^*.*

Theorem 36. *Let $L(G) = (\mu_2, \nu_2)$ be the $\mathcal{N}$-line graph corresponding to $\mathcal{N}$-graph $G = (\mu_1, \nu_1)$. Suppose that $G^* = (V, E)$ is connected. Then*

(1) *there exists a strong isomorphism of G onto $L(G)$ if and only if G^* is a cyclic and for all $v \in V$, $x \in E$, $\mu_1(v) = \nu_1(x)$, i.e., μ_1 and ν_1 are constant functions on V and E, respectively, taking on the same value.*

(2) *If f is a strong isomorphism of G onto $L(G)$, then f is an isomorphism.*

Proof. Assume that f is a strong isomorphism of G onto $L(G)$. From Proposition 3.31, it follows that $G^* = (V, E)$ is a cycle [12, Theorem 8.2, p.72]. Let $V = \{v_1, v_2, \cdots, v_n\}$ and $E = \{x_1 = v_1v_2, x_2 = v_2v_3, \cdots, x_n = v_nv_1\}$, where $v_1v_2v_3\cdots v_nv_1$ is a cyclic. Define $\mathcal{N}$-functions

$$\mu_1(v_i) = \acute{s}_i,\ \nu_1(v_iv_{i+1}) = \acute{r}_i, i = 1, 2, \cdots, n, v_{n+1} = v_1.$$

Then for $\acute{s}_{n+1} = \acute{s}_1$,

$$\text{(a) } \{\ \acute{r}_i \geq \max(\acute{s}_i, \acute{s}_{i+1}), i = 1, 2, \cdots, n.$$

Now

$$Z = \{S_{x_1}, S_{x_1}, S_{x_2}, \cdots, S_{x_n}\}$$

$$W = \{S_{x_1}S_{x_2}, S_{x_2}S_{x_3}, \cdots, S_{x_n}S_{x_1}\}.$$

Also for $r_{n+1} = r_1$,

$$\mu_2(S_{x_i}) = \mu_1(x_i) = \mu_1(v_iv_{i+1}) = \acute{r}_i,$$

$$\begin{aligned} \nu_2(S_{x_i}S_{x_{i+1}}) &= \max(\nu_1(x_i), \nu_1(x_{i+1})) \\ &= \max(\mu_1(v_iv_{i+1}), \nu_1(v_{i+1}v_{i+2})) \\ &= \max(\acute{r}_i, \acute{r}_{i+1}) \end{aligned}$$

for $i = 1, 2, \cdots, n$, $v_{n+1} = v_1$, $v_{n+2} = v_2$. Since f is an isomorphism of G^* onto $L(G^*)$, f maps V one-to-one and onto Z. Also f preserves adjacency. Hence f induces a permutation π of $\{1, 2, \cdots, n\}$ such that

$$f(v_i) = S_{x_{\pi(i)}} = S_{x_{\pi(i)}}S_{x_{\pi(i+1)}}$$

and

$$\begin{aligned} x_i = v_iv_{i+1} \to f(v_i)f(v_{i+1}) &= S_{v_{\pi(i)}}S_{v_{\pi(i+1)}}S_{v_{\pi(i+2)}} \\ &,\quad i = 1, 2, \cdots, n-1. \end{aligned}$$

Now

$$\acute{s}_i = \mu_1(v_i) \geq \mu_2(f(v_i)) = \mu_2(S_{v_{\pi(i)}v_{\pi(i+1)}}) = \acute{r}_{\pi(i)},$$

$$\begin{aligned} \acute{r}_i = \nu_1(v_iv_{i+1}) &\geq \nu_2(f(v_i)f(v_{i+1})) \\ &= \nu_2(S_{v_{\pi(i)}}S_{v_{\pi(i)+1}}S_{v_{\pi(i+1)+1}}) \\ &= \max(\nu_1(v_{\pi(i)}v_{\pi(i)+1}), \nu_1(v_{\pi(i)+1}v_{\pi(i+1)+1})) \\ &= \max(\acute{r}_{\pi(i)}, \acute{r}_{\pi(i+1)}) \end{aligned}$$

for $i = 1, 2, \cdots, n$. That is,

$$\acute{s}_i \geq \acute{r}_\pi(i)$$

and

$$\text{(b) } \{\ \acute{r}_i \geq \max(\acute{r}_{\pi(i)}, \acute{r}_{\pi(i+1)}).$$

By (b), we have $\acute{r}_i \geq \acute{r}_{\pi(i)}$ for $i = 1, 2, \cdots, n$ and so $\acute{r}_\pi(i) \leq \acute{r}_{\pi(\pi(i))}$ for $i = 1, 2, \cdots, n$. Continuing, we have

$$\acute{r}_i \geq \acute{r}_{\pi(i)} \geq \cdots \geq \acute{r}_{\pi^j(i)} \geq \acute{r}_i$$

and so $r_i = r_{\pi(i)}$, $\acute{r}_i = \acute{r}_{\pi(i)}$, $i = 1, 2, \cdots, n$, where π^{j+1} is the identity map. Again, by (b), we have

$$\acute{r}_i \geq \acute{r}_{\pi(i+1)} = \acute{r}_{i+1}, i = 1, 2, \cdots, \acute{r}_{n+1} = \acute{r}_1.$$

Hence by (a) and (b),

$$\acute{r}_1 = \cdots = \acute{r}_n = \acute{s}_1 = \cdots = \acute{s}_n.$$

Thus we have not only proved the conclusion about μ_1 and ν_1 being constant function, but we have also shown that (2) holds. The converse part is obvious. □

We state the following Theorem without proof.

Theorem 37. *Let G and H be $\mathcal{N}$-graphs of G^* and H^*, respectively, such that G^* and H^* are connected. Let $L(G)$ and $L(H)$ be the $\mathcal{N}$-line graphs corresponding to G and H, respectively. Suppose that it is not the case that one of G^* and $H*$ is complete graph K_3 and other is bipartite complete graph $K_{1,3}$. If $L(G)$ and $L(H)$ are isomorphic, then G and H are line-isomorphic.*

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